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On the Plasma Cerenkov Radiation II

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This work has been partially supported by NASA Grant NsG 220-62.

I. Introduction

In an earlier report[1] (hereinafter referred to as I) the problem of the field induced by a test particle injected into an equilibrium plasma was studied on the basis of the moment equations, with special attention to the effect of finite particle size and "transient" behavior. In the present work, we attack the same problem on the basis of the Vlasov equation, restricting ourselves, however, to the case of a point particle. The principal qualitative differences may be summarized as follows: (1) For slow (compared to thermal speed) particles, the leading correction to Debye-Hückel has opposite sign in front of and behind the particle; furthermore it <u>changes</u> sign at about 4.5 Debye lengths ahead of and behind the particle. Moreover, the amount of induced charge in front of the particle proves to be slightly less than behind, and there is a "ripple" in the polarization cloud. The symmetric flattening of the polarization cloud predicted by the moment equations is a higher order (in the Mach number) effect. Inasmuch as, for superthermal particles <u>all</u> the charge is behind the particle, it is not surprising that the first effect of finite particle speed is for the particle to "lead" its polarization cloud slightly. The physical origin of the "ripple" is not yet understood. (2) For fast particles, there is no singularity in the charge density or potential at the Mach cone, and there is some charge <u>outside</u> the cone. Thus the wake of the particle is somewhat "smeared out", and lacks the geometric sharpness predicted by the moment equations. Also, the potential near the cone is exponentially damped along the cone and dies out in a distance of about a Debye length times the Mach number. Well inside the cone, however, the decay of the wake is considerably slower.

All of these effects are due to the imaginary part of the dielectric function, and are thus related to the Landau damping; this explains their absence from the moment equation theory.

In the next section we derive an expression for the potential induced by a particle of charge q, velocity $\underset{\sim}{u}$, injected into an infinite equilibrium plasma at $t = 0$, assuming that the system is adequately described by the Vlasov and Maxwell equations. The problem is thus reduced to one of approximate evaluation of certain integrals. These are carried out in Section 3 (sub-thermal particles) and Section 4 (superthermal particles), with accompanying discussion. Section 5 is devoted to additional discussion and comparison with previous treatments.

II. Formulation of the Problem

We wish to calculate the field induced by a particle of charge q, velocity $\underset{\sim}{u}$ injected into a plasma initially at equilibrium. The plasma is assumed to consist of an arbitrary number of types of particle, where type σ has charge e_σ, mass m_σ, density n_σ. It is further assumed that the evolution of the system is adequately described by the Vlasov equation (the validity of this assumption will be discussed in Section 5). For simplicity we will consider only longitudinal fields, and neglect the reaction of the plasma on the test particle (the generalization to include transverse fields is straightforward).

If the perturbation of the one-particle distribution function for type σ is

$$f(\underset{\sim}{R},\underset{\sim}{v},\sigma,t) \equiv f(\underset{\sim}{R},\eta,t) \qquad ,$$

and the test particle is initially at $\underset{\sim}{R} = 0$, one has

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{R}} f + \frac{e_\sigma}{m_\sigma} \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_o(\eta) = 0 \qquad , \qquad (1)$$

$$\frac{\partial}{\partial \underset{\sim}{R}} \cdot \underset{\sim}{E}(\underset{\sim}{R},t) = 4\pi\left[\int d\eta e_\sigma f(\underset{\sim}{R},\eta,t) + q\, \delta(\underset{\sim}{R} - \underset{\sim}{u}t)\right] \qquad , \qquad (2)$$

$$\frac{\partial}{\partial \underset{\sim}{R}} \times \underset{\sim}{E} = 0 \qquad , \qquad (3)$$

where

$$\int d\eta \equiv \int d\underset{\sim}{v} \sum_\sigma$$

We Fourier-Laplace transform (1)-(3) by

$$F(\eta,\underset{\sim}{k},\omega) = \int_o^\infty dt \int d\underset{\sim}{R} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{R}-\omega t)} f(\underset{\sim}{R},\eta,t) \qquad , \qquad (4)$$

$$\underset{\sim}{\mathcal{E}}(\underset{\sim}{k},\omega) = \int_o^\infty dt \int d\underset{\sim}{R} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{R}-\omega t)} \underset{\sim}{E}(\underset{\sim}{R},t) \qquad , \qquad (5)$$

whereupon ($f(\underset{\sim}{R},\eta,0)$ is zero by assumption)

$$F = \frac{-\dfrac{ie_\sigma}{m_\sigma} \underset{\sim}{\mathcal{E}} \cdot \dfrac{\partial}{\partial \underset{\sim}{v}} f_o}{\underset{\sim}{k} \cdot \underset{\sim}{v} - \omega} \qquad , \qquad (6)$$

$$\underset{\sim}{k} \cdot \underset{\sim}{\mathcal{E}} = 4\pi\left[-\frac{q}{\underset{\sim}{k} \cdot \underset{\sim}{u} - \omega} + i \int d\eta e_\sigma F(\eta,\underset{\sim}{k},\omega)\right] \qquad , \qquad (7)$$

$$\underset{\sim}{\mathcal{E}} = \frac{\underset{\sim}{k}\varepsilon}{k} \qquad , \qquad (8)$$

where these expressions are valid for Imω < 0. Equation (8) may be used to express (6) and (7) in terms of ε; multiplying (6) by e_σ, integrating over η and substituting into (7), one finds

$$\varepsilon(\underset{\sim}{k},\omega) = -\frac{4\pi(q/k)}{(\underset{\sim}{k}\cdot\underset{\sim}{u}-\omega)\Delta^-(\omega,\underset{\sim}{k})} \quad , \tag{9}$$

with

$$\Delta^-(\omega,\underset{\sim}{k}) = 1 - \frac{4\pi}{k^2}\int \frac{d\eta e_\sigma{}^2\underset{\sim}{k}\cdot\frac{\partial}{\partial\underset{\sim}{v}} f_o(\eta)}{m_\sigma(\underset{\sim}{k}\cdot\underset{\sim}{v}-\omega)} \tag{10}$$

Taking the inverse transforms, one has

$$\underset{\sim}{E}(\underset{\sim}{R},t) = \frac{1}{(2\pi)^4}\int d\underset{\sim}{k}\int_{-\infty-i\gamma}^{\infty-i\gamma} d\omega e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{R}-\omega t)}\frac{\underset{\sim}{k}}{k}\varepsilon(\underset{\sim}{k},\omega) =$$

$$\frac{q}{4\pi^3}\int d\underset{\sim}{k}\frac{\underset{\sim}{k}e^{-i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{k^2}\int_{-\infty-i\gamma}^{\infty-i\gamma} d\omega\frac{e^{i\omega t}}{(\omega-\underset{\sim}{k}\cdot\underset{\sim}{u})\Delta^-(\omega,\underset{\sim}{k})} \tag{11}$$

The properties of the dielectric function Δ^- are well known; denoting its zeroes as a function of ω by $\omega_n(\underset{\sim}{k})$, one may do the ω integration by closing the contour in the upper half-plane to obtain

$$\underset{\sim}{E}(\underset{\sim}{R},t) = -\nabla\phi(\underset{\sim}{R},t) \quad ;$$

$$\phi(\underset{\sim}{R},t) = \frac{q}{2\pi^2}\int\frac{d\underset{\sim}{k}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{k^2}\left[\frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{u}t}}{\Delta^-(\underset{\sim}{k}\cdot\underset{\sim}{u},\underset{\sim}{k})}\right.$$

$$\left. + \sum_n \frac{e^{i\omega_n(\underset{\sim}{k})t}}{(\omega_n(\underset{\sim}{k})-\underset{\sim}{k}\cdot\underset{\sim}{u})\left(\frac{\partial\Delta^-(\omega,\underset{\sim}{k})}{\partial\omega}\right)_{\omega\to\omega_n(\underset{\sim}{k})}}\right] \tag{12}$$

Equation (12) is the formal solution of the problem under the given assumptions; the remainder of the report is devoted to the evaluation and interpretation of (12) for the limiting cases of small and large u, where analytical expressions may be obtained.

III. Subthermal Particles; Correction to the Debye Potential

We first consider the case where the test particle speed u is much smaller than the thermal velocity of any of the components. Using

$$f_o(\eta) = \frac{n_\sigma e^{-\frac{v^2}{V_\sigma^2}}}{(\sqrt{\pi}\, V_\sigma)^3} \quad , \tag{13}$$

$$\frac{1}{x + i\epsilon} = P\left(\frac{1}{x}\right) - \pi i \delta(x) \quad , \tag{14}$$

and

$$P \int_{-\infty}^{\infty} \frac{dy\; e^{-\alpha y^2}}{y - x} = -2\sqrt{\pi}\, e^{-\alpha x^2} \int_0^{x\sqrt{\alpha}} dy\; e^{y^2} \quad , \tag{15}$$

we readily reduce (10) to the familiar form

$$\Delta^-(\omega,\underset{\sim}{k}) = 1 + \sum_\sigma \frac{k_\sigma^2}{k^2} \left\{ 1 - \frac{\omega}{kV_\sigma} e^{-\frac{\omega^2}{k^2 V_\sigma^2}} \left[\sqrt{\pi}\, i + 2 \int_0^{\frac{\omega}{kV_\sigma}} dx\; e^{x^2} \right] \right\} \quad , \tag{16}$$

where

$$k_\sigma^{\,2} = 8\pi \frac{n_\sigma e_\sigma^{\,2}}{m_\sigma V_\sigma^{\,2}} \tag{17}$$

Now if $u \ll V_\sigma$ for all σ,

$$(\underset{\sim}{k}\cdot\underset{\sim}{u}/kV_\sigma) \leq u/V_\sigma \ll 1,$$

and we may write

$$\Delta^-(\underset{\sim}{k}\cdot\underset{\sim}{u},\underset{\sim}{k}) = 1 + \sum_\sigma \frac{k_\sigma{}^2}{k^2}\left\{1 - \frac{\sqrt{\pi}\,i\underset{\sim}{k}\cdot\underset{\sim}{u}}{kV_\sigma} - 2\left(\frac{\underset{\sim}{k}\cdot\underset{\sim}{u}}{kV_\sigma}\right)^2 + 0\left(\frac{u^3}{V_\sigma{}^3}\right)\right\}. \tag{17a}$$

Then the first term of (12), which we denote by ϕ_1, may be written in spherical coordinates, with $(\underset{\sim}{R} - \underset{\sim}{u}t)$ taken as the z-axis, as

$$\phi_1(\underset{\sim}{R},t) = \frac{q}{2\pi^2}\int_0^{2\pi} d\phi \int_{-1}^{1} d\mu \int_0^\infty dk\; e^{-ik|\underset{\sim}{R}-\underset{\sim}{u}t|\mu}\left\{\frac{1}{1+\frac{K^2}{k^2}} + \frac{\sqrt{\pi}i}{\left(1+\frac{K^2}{k^2}\right)^2}\sum_\sigma \frac{k_\sigma^2}{k^2}\left(\frac{u}{V_\sigma}\right)\left[\mu\mu_o\right.\right.$$

$$\left. + \sqrt{1-\mu^2}\,\sqrt{1-\mu_o{}^2}\,\cos\phi\right] + \frac{1}{\left(1+\frac{K^2}{k^2}\right)^2}\left[\mu\mu_o + \sqrt{1-\mu^2}\,\sqrt{1-\mu_o{}^2}\,\cos\phi\right]^2 \times$$

$$\left.\left[2\sum_\sigma \frac{k_\sigma{}^2}{k^2}\frac{u^2}{V_\sigma{}^2} - \frac{\pi}{\left(1+\frac{K^2}{k^2}\right)}\left(\sum_\sigma \frac{k_\sigma{}^2}{k^2}\frac{u}{V_\sigma}\right)^2\right] + 0\left(\frac{u^3}{V^3_{min}}\right)\right\}, \tag{18}$$

where

$$K^2 = \sum_\sigma k_\sigma{}^2, \quad \mu_o = \frac{\underset{\sim}{u}\cdot(\underset{\sim}{R}-\underset{\sim}{u}t)}{u|\underset{\sim}{R}-\underset{\sim}{u}t|} \tag{19}$$

The angular integrals are straightforward, and one finds

$$\phi_1(\underset{\sim}{R},t) = \frac{2q}{\pi|\underset{\sim}{R}-\underset{\sim}{u}t|}\int_0^\infty \frac{dk\;k}{(k^2+K^2)}\left\{\sin k|\underset{\sim}{R}-\underset{\sim}{u}t|\right.$$

$$- \sqrt{\pi} \sum_\sigma \frac{k_\sigma^{\,2} u}{V_\sigma} \quad \frac{\mu_o}{k^2 + K^2} \left[\cos k|\underset{\sim}{R}-\underset{\sim}{u}t| - \frac{\sin k|\underset{\sim}{R}-\underset{\sim}{u}t|}{k|\underset{\sim}{R}-\underset{\sim}{u}t|} \right]$$

$$+ \frac{1}{(k^2 + K^2)} \left[\mu_o^{\,2} \left(\sin k|\underset{\sim}{R}-\underset{\sim}{u}t| + \frac{2\cos k|\underset{\sim}{R}-\underset{\sim}{u}t|}{k|\underset{\sim}{R}-ut|} \right. \right.$$

$$\left. \left. - \frac{2\sin k|\underset{\sim}{R}-\underset{\sim}{u}t|}{(k|\underset{\sim}{R}-\underset{\sim}{u}t|)^2} \right) \quad + (1 - \mu_o^{\,2}) \left(\frac{\sin k|\underset{\sim}{R}-\underset{\sim}{u}t|}{(k|\underset{\sim}{R}-\underset{\sim}{u}t|)^2} - \frac{\cos k|\underset{\sim}{R}-\underset{\sim}{u}t|}{k|\underset{\sim}{R}-\underset{\sim}{u}t|} \right) \right]$$

$$\left[2 \sum_\sigma k_\sigma^{\,2} \frac{u^2}{V_\sigma^{\,2}} - \frac{\pi}{(k^2 + K^2)} \left(\sum_\sigma k_\sigma^{\,2} \frac{u}{V_\sigma} \right)^2 \right] + 0 \left(\frac{u^3}{V^3_{min}} \right) \qquad . \tag{20}$$

The k integral in the first term of (20) may be done by contour integration and leads to the familiar Debye-Hückel result for the polarization cloud about a stationary particle*

$$\phi_1^{(o)} = \frac{q\, e^{-K|\underset{\sim}{R}-\underset{\sim}{u}t|}}{|\underset{\sim}{R}-\underset{\sim}{u}t|} \tag{21}$$

The leading correction to the Debye result for slow particles is given by the second term of (20). This term, first order in the

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* At this point we should remark the fact, usually disguised by a different definition of K, that the moment equation treatment gives a Debye length which is off by a factor of $\sqrt{3}$.

particle speed, cannot be obtained from the collision-less moment equations, as it comes from the imaginary part of the dielectric function Δ^{-}. Introducing $z \equiv k|\underset{\sim}{R}-\underset{\sim}{u}t|$,

$$\alpha = \frac{1}{\sqrt{\pi}} \sum_{\sigma} \frac{k_{\sigma}{}^{2} u}{K^{2} V_{\sigma}} \quad ,$$

$$\underset{\sim}{\lambda} = K(\underset{\sim}{R}-\underset{\sim}{u}t), \quad ,$$

$$\beta = \sum_{\sigma} k_{\sigma}{}^{2}\ u^{2}/K^{2} V_{\sigma}{}^{2}, \tag{22}$$

one may write the second term of (20) as

$$\phi_{1}{}^{(1)} = qK\alpha \frac{\underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{u\lambda} \frac{\partial}{\partial\lambda} L(\lambda) \quad , \tag{23}$$

where

$$L(\lambda) = \int_{0}^{\infty} \frac{dz}{z^{2}+\lambda^{2}} (z \cos z - \sin z) \tag{24}$$

After some algebra and deforming of integration paths, one can express (24) in terms of exponential integrals. One then finds

$$L(\lambda) = -\frac{1}{2}\left[\left(1-\frac{1}{\lambda}\right) e^{\lambda}\, \mathcal{E}i(-\lambda) + \left(1+\frac{1}{\lambda}\right) e^{-\lambda}\, \overline{\mathcal{E}i}(\lambda)\right] \quad , \tag{25}$$

where

$$\mathcal{E}i(-\lambda) = -\int_{\lambda}^{\infty} \frac{dy\ e^{-y}}{y} \quad , \tag{26}$$

$$\overline{\mathcal{E}i}(\lambda) = -P\int_{-\lambda}^{\infty} \frac{dy\ e^{-y}}{y} \tag{27}$$

It follows that

$$\phi_1^{(1)} = \frac{qK\ \alpha\ \underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{u\lambda}\left[-\frac{1}{\lambda}-\left(1-\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)\frac{e^{\lambda}\mathcal{E}i(-\lambda)}{2}+\left(1+\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)\frac{e^{-\lambda}\overline{\mathcal{E}i}(\lambda)}{2}\right]$$

$$\equiv \frac{qK\ \alpha\ \underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{u\lambda}\ M(\lambda) \tag{28}$$

It is convenient to calculate the charge density corresponding to (28), by

$$\rho_1^{(1)} = -\frac{K^2}{4\pi}\ \nabla^2_{\underset{\sim}{\lambda}}\ \phi_1^{(1)} = -\frac{qK^3\alpha\underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{4\pi\ u\lambda}\left(\frac{d^2M}{d\lambda^2}+\frac{2}{\lambda}\frac{dM}{d\lambda}-\frac{2}{\lambda^2}M\right)$$

$$= \frac{qK^3\alpha\underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{8\pi\ u\lambda^3}\left[2\lambda+(\lambda^2+\lambda-1)\ e^{+\lambda}\mathcal{E}i(-\lambda)-(\lambda^2-\lambda-1)\ e^{-\lambda}\overline{\mathcal{E}i}(\lambda)\right] \quad . \tag{29}$$

As $\lambda \to 0$, one has

$$\mathcal{E}i(-\lambda) = (\gamma + \log\lambda) + O(\lambda),\ \lambda \ll 1 \quad , \tag{30}$$

$$\overline{\mathcal{E}i}(\lambda) - \mathcal{E}i(-\lambda) = 2\lambda(1 + O(\lambda^2)),\ \lambda \ll 1 \quad , \tag{31}$$

whence

$$\rho_1^{(1)} \cong \frac{qK^3\alpha\underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{2\pi\ u\lambda^2},\quad \lambda \ll 1 \tag{32}$$

It will be observed that (32) is small (because of the small factor α as defined in Eqn. (22) compared to the polarization charge $-\frac{qK^3e^{-\lambda}}{4\pi\ \lambda}$ from (21), but has <u>opposite</u> sign in front of the particle and the <u>same</u> sign behind. Thus the compensating charge near the particle is reduced ahead of the particle and increased behind, i.e. the particle is slightly "ahead" of its polarization cloud.

On the other hand, for large λ, one has

$$e^{\lambda}\,\mathcal{E}i(-\lambda) \approx -\frac{1}{\lambda}\left(1 - \frac{1}{\lambda} + \frac{2}{\lambda^2} - \frac{6}{\lambda^3} + \frac{24}{\lambda^4}\right), \quad \lambda >> 1 \quad , \tag{33}$$

and

$$e^{-\lambda}\,\overline{\mathcal{E}i}(\lambda) \approx \frac{1}{\lambda}\left(1 + \frac{1}{\lambda} + \frac{2}{\lambda^2} + \frac{6}{\lambda^3} + \frac{24}{\lambda^4}\right) + 0\left(\frac{e^{-\lambda}}{\lambda}\right), \quad \lambda >> 1 \;. \tag{34}$$

It follows that

$$\rho_1^{(1)} = -\frac{4qK^3\alpha \underset{\sim}{u}\cdot\underset{\sim}{\lambda}}{\pi u\lambda^6} \quad , \lambda >> 1 \tag{35}$$

It is apparent from (32), (35), that the correction to the charge density changes sign at a distance of order a Debye length from the particle. The exponential integrals $\mathcal{E}i(-\lambda)$, $\overline{\mathcal{E}i}(\lambda)$ have been tabulated; the expression (29) is plotted in Fig. 1. It will be observed that the charge density correction on the axis changes sign at about 4.5 Debye lengths and has a (small) extremum at about 5.5 Debye lengths, after which it goes slowly ($\sim \lambda^{-5}$) to zero. Equation (29) has been obtained by Neufeld and Ritchie,[2] who did not, however, investigate its properties or discover the "ripple" apparent in Fig. 1 and (32) and (35).

In view of the sign change of $\rho_1^{(1)}$, an additional calculation is necessary to show that (29) does indeed represent a slight shift of polarization charge from the front to the rear of the particle. The total charge contribution in front of the particle from (29) is given by

$$Q_+ = \frac{1}{K^3}\int_{\underset{\sim}{u}\cdot\underset{\sim}{\lambda}>0} d\underset{\sim}{\lambda}\;\rho_1^{(1)}(\underset{\sim}{\lambda}) \tag{36}$$

The integral may be done with the help of a straightforward integration by parts, and the asymptotic forms (33), (34); one finds

$$Q_+ = \frac{q\alpha}{2} \qquad ,$$

which is opposite to the Debye polarization charge as expected. The charge behind the particle is given by $Q_- = -Q_+$, so it is evident that (29) gives no contribution to the total net polarization charge, but represents a slight shift of charge from the front to the rear of the particle.

One additional peculiarity of the correction $\rho_1^{(1)}$ should be mentioned; from the definitions (22) and (17), the largest contributions came from the slowest (and thus the most massive) ions.

We now turn to the last term of (20). The k integral may be done straightforwardly, if tediously, by contour integration, and one finds

$$\phi_1^{(2)} = \frac{qK}{2}\left\{\beta\left[e^{-\lambda}\left(\mu_o^2 - \frac{(1-3\mu_o^2)}{\lambda} - \frac{2(1-3\mu_o^2)}{\lambda^2} - \frac{2(1-3\mu_o^2)}{\lambda^3}\right) + \frac{2(1-3\mu_o^2)}{\lambda^3}\right] - \frac{\pi^2\alpha^2}{8}\left[e^{-\lambda}\left(\mu_o^2(\lambda+1) - (1-3\mu_o^2) - \frac{4(1-3\mu_o^2)}{\lambda} - \frac{8(1-3\mu_o^2)}{\lambda^2} - \frac{8(1-3\mu_o^2)}{\lambda^3}\right) + \frac{8(1-3\mu_o^2)}{\lambda^3}\right]\right\} \qquad (37)$$

with the corresponding charge density* (the quantities α and β are defined in (22))

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* Note that $\mu_o = \underset{\sim}{\lambda}\cdot\underset{\sim}{u}/(\lambda u)$ must be differentiated in calculating (38).

$$\rho_1^{(2)} = - \frac{K^2 \nabla_{\underset{\sim}{\lambda}}^2 \phi_1^{(2)}(\underset{\sim}{\lambda})}{4\pi} = \frac{qK^3}{8\pi} e^{-\lambda} \left\{ \beta \left[\frac{1 - \mu_o^2}{\lambda} - \mu_o^2 \right] \right.$$

$$\left. - \frac{\pi^2\alpha^2}{8} \left[1 - \mu_o^2 \lambda \right] \right\} \tag{38}$$

We make the following observations about (37) and (38): (a) The correction to the potential (but not the charge density correction) has a tail which is not exponentially damped, but goes as λ^{-3}. This tail is <u>opposite</u> in sign to the Debye potential near the path ($|\mu_o| < 1/\sqrt{3}$) and has the <u>same</u> sign near the mid-plane. (b) The charge density described by (38) is distributed as described in Fig. 2(a). Inasmuch as the Debye polarization cloud is negative (for positive q), we see that the polarization cloud is reduced in the outer region and the inner teardrop shaped regions marked (+), and enhanced in the region marked (-). Thus even this correction is not a simple flattening of the polarization cloud as predicted by the moment equations. In fact, one may demonstrate that the transfer of charge from the "teardrop" regions is somewhat larger than that from the outer region. If we assume that the slowest ion is much slower than the next slowest, so that $\beta \underset{\sim}{\sim} \pi\alpha^2$, one finds for the total charge in the "teardrop" regions

$$Q_{2i} = \int_{V_i} d\underset{\sim}{\lambda} \; \rho_1^{(2)}(\lambda) \cong \frac{qK^3\pi^2\alpha^2}{24} \int_0^{(8/\pi)} \frac{d\lambda \lambda e^{-\lambda} \left(\frac{8}{\pi} - \lambda \right)^{3/2}}{\sqrt{(8/\pi)(\lambda+1) - \lambda^2}} \cong .23qK^3\alpha^2 \tag{39}$$

where V_i is the "teardrop" region defined by

$$\lambda < (8/\pi), \; |\mu_o| < \sqrt{\frac{(8/\pi) - \lambda}{(8/\pi)(\lambda+1) - \lambda^2}} \quad ,$$

and the finite integral in (39) has been calculated approximately by numerical integration.

Similarly the charge in the "outer" region is given by

$$Q_{2o} = \int_{V_o} d\underset{\sim}{\lambda}\, \rho_1^{(2)}(\lambda) \cong \frac{qK^3\pi^2\alpha^2}{16} \int_{1+(8/\pi)}^{\infty} d\lambda\lambda e^{-\lambda}\left\{\frac{\lambda^2}{3} - \lambda\left(1 + \frac{8}{3\pi}\right)\right.$$

$$\left. + (16/3\pi) + (2/3)(\lambda - \gamma)^{3/2}[\lambda^2 - (8/\pi)(\lambda + 1)]^{-1/2}\right\} \cong .13qK^3\alpha^2 \qquad (40)$$

where the region V_o is defined by

$$\lambda > 1 + 8/\pi, \quad |\mu_o| > \sqrt{\frac{\lambda - (8/\pi)}{\lambda^2 - (8/\pi)(\lambda + 1)}}$$

(c) Equation (38) represents a redistribution of charge, not an addition or subtraction; the total amount of charge remains constant as can be shown by the direct calculation of

$$Q_2 = \int d\underset{\sim}{\lambda}\, \rho_1^{(2)}(\lambda) = 0$$

(d) The discrepancy between the results of I and the correction (38) is not entirely due to the neglect of the imaginary part of the dielectric function (unlike the first order correction (29) which is missed by the moment equation treatment). If we had neglected the imaginary part of Δ^-, we would have found a charge distribution like that depicted in Fig. 2(b) (which is the <u>opposite</u> of a flattening of the polarization cloud!). On the other hand, if one expands the slow particle result of I in powers of the Mach number, the first correction $(O(M^{-2}))$ to the Debye cloud resembles Fig. 2(b) except that the Debye length is off by a factor of $\sqrt{3}$ and the charge correction has opposite sign. This can be understood noting that the neglect of the imaginary part of Δ^- (cf. (17)) leads, for an electron gas in a positive background, to

$$\frac{1}{\Delta^-(\underset{\sim}{k}\cdot\underset{\sim}{u},k)} - 1 \to -\frac{K^2}{k^2+K^2} + \frac{2K^2(\underset{\sim}{k}\cdot\underset{\sim}{u})^2}{(k^2+K^2)^2 V_e^{\,2}} + O(M^4) \quad ,$$

whereas the moment equations (compare the first term of (12) with (I.32)) give, in our notation

$$\frac{1}{\Delta^-(\underset{\sim}{k}\cdot\underset{\sim}{u},k)} - 1 \to -(K^2/3)\left[\frac{1}{k^2+K^2/3} + \frac{2(\underset{\sim}{k}\cdot\underset{\sim}{u})^2}{3V_e^{\,2}(k^2+K^2/3)^2}\right] + O(M^4) \quad .$$

Thus it is apparent that the moment equations give the wrong sign on the $O(M^2)$ correction to the real part of the dielectric function.

From the above, we are forced to conclude that the moment equations lead to results for slow particles which are qualitatively incorrect in almost every detail (a fact which should not greatly surprise anyone who has tried to estimate the range of validity of the moment equation treatment).

Finally, we turn to the "transient" contribution given by the second term of (12). In view of the exponential factor, we may restrict ourselves to $k \ll K$. (The contributions neglected thereby will be damped exponentially in time with a lifetime of order a plasma period. It will be recalled that we are assuming Maxwellian f, and thus need not worry about growing roots of $\Delta^-(\omega,\underset{\sim}{k}) = 0$.) For simplicity we consider a two-component system in which the ions and electrons have the same temperatures and the ions are much heavier than the electrons. Then the least damped roots of $\Delta^-(\omega,\underset{\sim}{k}) = 0$ are given approximately by the Landau poles

$$\omega_{\pm}(k) = \pm\, \Omega(k) + i\gamma(k) \tag{41}$$

where

$$\Omega^2(k) = \omega_o^{\,2} \left(1 + \frac{6k^2}{K^2}\right) \quad , \tag{42}$$

$$\gamma(k) = \frac{\omega_o \sqrt{\pi e^{-3}} K^3 \, e^{-\frac{K^2}{4k^2}}}{8k^3} \quad , \tag{43}$$

with

$$\omega_o \equiv \sqrt{4\pi \sum_\sigma \frac{n_\sigma e_\sigma^{\,2}}{m_\sigma}} \quad , \tag{44}$$

and K is given by (19). Equations (42), (43) differ slightly from the usual forms because, from the outset, we have taken K to be the <u>total</u> Debye wave number, not just the <u>electron</u> contribution (observe that electrons and singly charged ions of the same temperature give equal contributions to (19)). In deriving (42), (43) we have neglected terms proportional to the mass ratios which could easily be included and would not change the general character of the result. In view of (41)-(43) we have

$$\left|\frac{\underset{\sim}{k} \cdot \underset{\sim}{u}}{\omega_{\pm}}\right| \leq \frac{ku}{|\omega_{\pm}|} \approx \frac{ku}{\omega_o} \approx \sqrt{2}\left(\frac{k}{K}\right) \frac{u}{V_e} << 1 \tag{45}$$

so that the denominator $(\omega_n - \underset{\sim}{k}\cdot\underset{\sim}{u})^{-1}$ in the second term of (12) may be expanded, and we will keep only the first two terms (the δ-function contribution is, of course, negligibly small). Furthermore

$$\left.\frac{\partial \Delta^-(\omega,k)}{\partial \omega}\right|_{\omega \to \omega_\pm(k)} \approx \pm \frac{2\omega_o^{\,2}}{\Omega(k)^3} \tag{46}$$

It follows that we may write the second term of (12) approximately as

$$\phi_2 = \frac{2qK}{\pi L}\int_0^\infty d\mathfrak{z}\; e^{-\Gamma(\mathfrak{z})T}\left\{\left(\frac{1+6\mathfrak{z}^2}{\mathfrak{z}}\right)\sin L\mathfrak{z}\,\cos T(1+3\mathfrak{z}^2) + \right.$$

$$\left.\frac{2\underset{\sim}{L}\cdot\underset{\sim}{u}}{L^2V_e}\sqrt{1+3\mathfrak{z}^2}\left[-\cos L\mathfrak{z}\,\sin T(1+3\mathfrak{z}^2) + \frac{\sin L\mathfrak{z}\,\sin T(1+3\mathfrak{z}^2)}{L\mathfrak{z}}\right]\right\} \tag{47}$$

where

$$\mathfrak{z} = k/K, \quad \underset{\sim}{L} = K\underset{\sim}{R}, \quad T = \omega_o t \quad , \tag{48}$$

and

$$\Gamma(k/K) = \frac{\gamma(k)}{\omega_o} \quad , \tag{49}$$

i.e. Γ is γ made dimensionless and expressed in terms of the dimensionless wave number. Note that the distance $\underset{\sim}{R}$ is the distance from the <u>initial</u> position of the particle, not from the present position of the particle (which is $\underset{\sim}{R} - \underset{\sim}{u}t$). Thus in the neighborhood of the particle,

$$L \underset{\sim}{\sim} Kut << T \underset{\sim}{\sim} KV_e t \tag{50}$$

Now for $\mathfrak{z} \gtrsim 1$, $\Gamma(\mathfrak{z}) \gtrsim 1$, so contributions from this region will be exponentially damped. On the other hand, for $\mathfrak{z} << 1$, $\Gamma(\mathfrak{z}) \to 0$; therefore we may estimate (47) by replacing the limits by (0,1) and dropping the factor $e^{-\Gamma T}$. It should be emphasized that there are terms which go as e^{-T}; however, the main contribution will be shown to decay much more slowly.

The form of (47) strongly suggests that the result may be expressed in terms of Fresnel integrals, and this is indeed the case; but the resulting expression is so long as to be unwieldy. However, two important limiting cases can be approximated without difficulty. Firstly, if $T \gg L^2$, the rapid oscillations of the factors $\cos T(1+3\zeta^2)$, $\sin T(1+3\zeta^2)$, insure that the main contribution to the integrand comes from $\zeta \lesssim (3T)^{-1/2}$, so that $L\zeta \lesssim \frac{L}{\sqrt{3T}} \ll 1$. It follows that we can let $\zeta \to 0$, except in the rapidly oscillating factors to obtain

$$\phi_2 \approx \frac{2qK}{\pi} \int_0^1 d\zeta \left\{ \cos T\,(1+3\zeta^2) + \frac{2}{3}\frac{\underset{\sim}{L}\cdot\underset{\sim}{u}}{LV_e}\zeta^2 \sin T\,(1+3\zeta^2) \right\}, \quad \begin{matrix} T \gg L^2, \\ T \gg 1 \end{matrix}. \tag{51}$$

But

$$\int_0^1 d\zeta \cos T\,(1+3\zeta^2) = \frac{1}{\sqrt{3T}} \int_0^{\sqrt{3T}} du\,[\cos T \cos u^2 - \sin T \sin u^2]. \tag{52}$$

For large T, we may replace the upper limit by ∞ and use

$$\int_0^\infty du \cos u^2 = \int_0^\infty du \sin u^2 = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{53}$$

to obtain

$$\int_0^1 d\zeta \cos T\,(1+3\zeta^2) \approx \frac{1}{2}\sqrt{\frac{\pi}{6T}}\,(\cos T - \sin T), \quad T \gg 1 \tag{54}$$

Similarly

$$\int_0^1 d\zeta\,\zeta^2 \sin T(1+3\zeta^2) = \frac{1}{4T}\sqrt{\frac{\pi}{6T}}\,\left[\cos T - \sin T\right], \quad T \gg 1 . \tag{55}$$

It follows that

$$\phi_2 = \frac{qK}{\sqrt{6\pi T}} (\cos T - \sin T) \left[1 + 0\left(\frac{1}{T}\right) + 0\left(\frac{L^2}{T}\right)\right] \quad , \quad T >> L^2, 1 \quad . \tag{56}$$

On the other hand, for $L >> \sqrt{T} >> 1$, the oscillations of the form $\binom{\sin}{\cos}(3\mathfrak{z}^2 T)$ may be ignored and one finds

$$\phi_2 \approx \frac{qK}{L} \cos T \left[1 + 0\left(\frac{1}{L}\right) + 0\left(\frac{T}{L^2}\right)\right] \quad , \quad L >> \sqrt{T} >> 1 \tag{57}$$

From (56), (57) it is evident that the "transient" contribution will die out at least as fast as $(\omega_o t)^{-1/2}$, independent of R. To be sure, this is not a very rapid decay, and it appears that, even neglecting boundary effects, one may have to wait many plasma periods for the onset of the "steady state."

This completes the study of the "slow particle" problem. It will be noted that we have found a number of qualitative differences from the usual moment equation treatment. Since we have assumed $u << V_{min}$, where V_{min} is the thermal velocity of the slowest ion, it is doubtful whether these differences can be tested experimentally. (A treatment of the region $V_i << u << V_e$, which is slightly more difficult, has been given by Kraus and Watson.[3]) It should be pointed out that the qualitative differences found here also occur in the common (but somewhat unrealistic) model of an electron gas in a "smeared out" positive background.

IV. Superthermal Particles

We now turn to the case where the injected particle is travelling much faster than the thermal speed of any of the components of the plasma. In this case, it is convenient to write the first term of (12) in cylindrical coordinates with $\underset{\sim}{u}$ as the z-axis.

$$\phi_1 = \frac{q}{2\pi^2}\int_0^{2\pi} d\phi \int_0^{\infty} dk_\perp k_\perp \int_{-\infty}^{\infty} \frac{dk_z \, e^{i[k_z(ut-z)-k_\perp R_\perp \cos\phi]}}{(k_z^2 + k_\perp^2)\ \Delta^-(k_z u, \sqrt{k_z^2 + k_\perp^2}\,)} \quad . \tag{58}$$

We propose to do the k_z integration by closing the contour in the upper half plane for $ut > z$ (behind the particle) or in the lower half plane for $ut < z$. The following properties of $\Delta^-(k_z u, \sqrt{k_z^2 + k_\perp^2}$ in the complex k_z plane may be deduced from (10) or (16): (a) It has branch points at $k_z = \pm\, ik_\perp$, with branch cuts which may be taken to run from $ik_\perp$ to $i\infty$ and $(-ik_\perp)$ to $(-i\infty)$. (b) In the lower half plane of k_z, Δ^- is analytic except for the branch cut, has no zeroes, and approaches unity on the large and small circles $|k_z| \to \infty$ and $k_z \to -ik_\perp$. (c) In the upper half plane Δ^- has an infinite number of zeroes, approaches unity as $|k_z| \to \infty$, and approaches either 0 or ∞ as $k_z \to ik_\perp$, depending on the direction of approach (the exact behavior will be shown later).

We first discuss the potential in front of the particle, $z > ut$. In view of (b) above, the integration from $(-\infty,\infty)$ may be replaced by an integral over the path C of Fig. 3. The integral over the large circle clearly vanishes; to evaluate the integrals around the branch cut and the branch point (which is also a pole) we need to consistently define the phase of $(k_z^2 + k_\perp^2)^{1/2}$. We first note that Δ^- may also be written as

$$\Delta^{-}(k_z u, \sqrt{k_z^2 + k_\perp^2}) = \mathcal{D}^{-}\left(\frac{k_z u}{(k_z^2 + k_\perp^2)^{1/2}}, k_z^2 + k_\perp^2\right) \tag{58a}$$

and $\mathcal{D}^{-}$ is analytic in the lower half plane of its first argument. Let us choose the phases of $k_z \pm ik_\perp$ in the following way:

$$k_z - ik_\perp = R_1 e^{i\phi_1}, \quad k_z + ik_\perp = R_2 e^{i\phi_2} \quad , \tag{59}$$

$$-\frac{3\pi}{2} < \phi_1 \leq \frac{\pi}{2} \quad , \quad -\frac{\pi}{2} < \phi_2 \leq \frac{3\pi}{2} \quad , \tag{60}$$

and $(k_z^2 + k_\perp^2)^{1/2}$ is defined as

$$(k_z^2 + k_\perp^2)^{1/2} = \sqrt{R_1 R_2}\, e^{\frac{i(\phi_1 + \phi_2)}{2}} \tag{61}$$

It is clear that this choice of phase locates the branch cuts as stated above. In fact, we may note the phase of $(k_z^2 + k_\perp^2)^{1/2}$ just to the left and right of the imaginary axis as follows

$$\left(\frac{\phi_1 + \phi_2}{2}\right)_L = -\frac{\pi}{2} \quad , \quad \left(\frac{\phi_1 + \phi_2}{2}\right)_R = \frac{\pi}{2} \quad , \quad \mathrm{Im} k_z > k_\perp \quad , \tag{62}$$

$$\left(\frac{\phi_1 + \phi_2}{2}\right)_L = \left(\frac{\phi_1 + \phi_2}{2}\right)_R = 0, \ -k_\perp < \mathrm{Im} k_z < k_\perp \quad , \tag{63}$$

$$\left(\frac{\phi_1 + \phi_2}{2}\right)_L = \frac{\pi}{2} \quad , \quad \left(\frac{\phi_1 + \phi_2}{2}\right)_R = -\frac{\pi}{2} \quad , \quad \mathrm{Im} k_z < (-k_\perp) \quad . \tag{64}$$

On the small circle in Fig. 3, $k_z = -ik_\perp + \epsilon\, e^{i\alpha}, \ -\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$ (65)

and the dielectric function is

$$\mathcal{D}^{-}\left(u\sqrt{\frac{k_\perp}{2\epsilon}}\, e^{-\frac{i}{2}\left(\alpha + \frac{\pi}{2}\right)}, -2ik_\perp \epsilon\, e^{i\alpha}\right) \quad , \tag{66}$$

where we have used (61) with

$$R_1 = 2k_\perp \quad , \quad R_2 = \epsilon$$

Since the first argument of (66) is in the lower half plane, we may use the asymptotic form

$$\lim_{\epsilon\to 0} \mathcal{D}^- \left(u\sqrt{\frac{k_\perp}{2\epsilon}} \; e^{-\frac{i}{2}\left(\frac{\pi}{2}+\alpha\right)} \; , \; -2ik_\perp\epsilon\, e^{i\alpha} \right) = 1 + \frac{\omega_o{}^2}{k_\perp u^2} \quad ,$$

where ω_o is given by Eq. (44). Denoting the contribution of the small circle by ϕ_{1_p}, we have

$$\phi_{1p} = \frac{q}{2\pi^2} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{k_\perp dk_\perp \; e^{-ik_\perp R_\perp \cos\phi}}{1 + \omega_o{}^2/k_\perp{}^2 u^2} \; e^{-k_\perp(\mathfrak{z}-ut)} \int_{\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{(i\epsilon\, e^{i\alpha})d\alpha}{(-2ik_\perp\epsilon\, e^{i\alpha})}$$

$$= \frac{q}{2\pi} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{k_\perp{}^2 dk_\perp \; e^{-k_\perp[\mathfrak{z}-ut+iR_\perp\cos\phi]}}{k_\perp{}^2 + \omega_o{}^2/u^2} \quad , \quad \mathfrak{z} > ut \tag{67}$$

The integrals in (67) may readily be approximated in the limits $|\underset{\sim}{R} - ut| >> u/\omega_o$ and $|\underset{\sim}{R} - ut| << u/\omega_o$. For the former case, the main contribution comes from $k_\perp << \omega_o/u$, and one finds

$$\phi_{1_p} = (qu^2/2\pi\omega_o{}^2) \int_0^{2\pi} d\phi \int_0^{\infty} dk_\perp k_\perp^2 \; e^{-k_\perp(\mathfrak{z}-ut+iR_\perp\cos\phi)} \left[1 + 0\left(u^2/\omega_o^2|\underset{\sim}{R}-\underset{\sim}{u}t|^2\right)\right]$$

$$= (qu^2/\pi\omega_o{}^2) \int_0^{2\pi} \frac{d\phi[1 + 0(u^2/\omega_o^2|\underset{\sim}{R}-\underset{\sim}{u}t|^2)]}{(\mathfrak{z}-ut + iR_\perp\cos\phi)^3}$$

$$= \left(qu^2/\omega_o^2|\underset{\sim}{R}-\underset{\sim}{u}t|^3\right) \left[\frac{3(\mathfrak{z}-ut)^2}{|\underset{\sim}{R}-\underset{\sim}{u}t|^2} - 1\right] \left[1 + 0\left(u^2/\omega_o^2|\underset{\sim}{R}-\underset{\sim}{u}t|^2\right)\right] \; ,$$

$$\omega_o|\underset{\sim}{R} - \underset{\sim}{u}t|/u >> 1 \tag{68}$$

On the other hand, in the opposite limit,

$$\phi_{1p} = \frac{q}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} dk_\perp \left[1 - \frac{\omega_o{}^2/u^2}{k_\perp^2 + \omega_o{}^2/u^2}\right] e^{-k_\perp L(\phi)}$$

$$= \frac{q}{|\mathbf{R}-\mathbf{u}t|} - \frac{q\omega_o}{4\pi i u}\int_0^{2\pi} d\phi \int_0^{\infty} dk_\perp\, e^{-k_\perp L(\phi)}\left[\frac{1}{k_\perp - i\omega_o/u} - \frac{1}{k_\perp + i\omega_o/u}\right]$$

$$= q\left\{\frac{1}{|\mathbf{R}-\mathbf{u}t|} - \frac{\omega_o}{4\pi i u}\int_0^{2\pi} d\phi \left[e^{-\frac{i\omega_o L(\phi)}{u}} \times \int_{-i\omega_o L(\phi)/u}^{\infty} dy\, e^{-y}/y - e^{\frac{i\omega_o L(\phi)}{u}} \int_{i\omega_o L(\phi)/u}^{\infty} dy\, e^{-y}/y\right]\right\}$$

$$= q\left\{\frac{1}{|\mathbf{R}-\mathbf{u}t|} - \frac{\omega_o}{2\pi u}\int_0^{2\pi} d\phi \left[\frac{\pi}{2} + \frac{\omega_o L(\phi)}{u}\left(\gamma - 1 + \log \omega_o L(\phi)/u\right) + 0(\omega_o^2 |L(\phi)|^2/u^2)\right]\right\}$$

$$= \frac{q}{|\mathbf{R}-\mathbf{u}t|}\left\{1 - \frac{\omega_o|\mathbf{R}-\mathbf{u}t|}{u}\left[\frac{\pi}{2} + \frac{\omega_o(\mathcal{Z}-ut)}{u}\left(\gamma - \frac{|\mathbf{R}-\mathbf{u}t|}{(\mathcal{Z}-ut)} + \log\frac{\omega_o}{2u}(\mathcal{Z}-ut+|\mathbf{R}-\mathbf{u}t|)\right)\right]\right.$$

$$\left. + 0(\omega_o{}^3|\mathbf{R}-\mathbf{u}t|^3/u^3)\right\}, \qquad \omega_o|\mathbf{R}-\mathbf{u}t|/u \ll 1 \qquad , \tag{69}$$

where we have defined $L(\phi) \equiv \mathcal{Z} - ut + iR_\perp\cos\phi$ (70)

and used the integrals

$$I_1 = \int_0^{2\pi} \frac{d\phi}{a + ib\cos\phi} = \frac{2\pi}{\sqrt{a^2 + b^2}} \qquad , \tag{71}$$

$$I_2 = \int_0^{2\pi} \frac{d\phi}{(a + ib\cos\phi)^3} = \frac{1}{2}\frac{\partial^2}{\partial a^2} I_1 = \frac{\pi}{(a^2+b^2)^{3/2}}\left[\frac{3a^2}{a^2+b^2} - 1\right] \quad , \tag{72}$$

$$I_3 = \int_0^{2\pi} d\phi(a + ib\cos\phi)\log(a + ib\cos\phi) = 2\pi\left\{a\left[1 + \log\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)\right] - \sqrt{a^2 + b^2}\right\} \tag{73}$$

Thus we see that the contribution to the potential near the particle is approximately the potential of the bare charge, while far from the particle there is a screening effect, leading to an R^{-3} behavior. (We shall demonstrate later that there is no induced charge in front of the particle.) The potential in front of the particle is due mainly to the integration around the small circle at the branch pole (Fig. 3) that we have just calculated, because, as we shall now show, the remaining contributions from the straight lines along the branch cut are negligible for a fast particle.

The contribution of the straight line paths in Fig. 3 can also be estimated without difficulty. Putting

$$k_{\mathfrak{z}} = -i(k_\perp + w) \qquad ,$$

using (61), (64), and calling this contribution ϕ_{1B}, one finds

$$\phi_{1B} = \frac{iq}{2\pi^2}\int_0^{2\pi} d\phi \int_0^\infty dk_\perp k_\perp \, P\int_0^\infty \frac{dw\, e^{-ik_\perp R_\perp\cos\phi}}{w(w + 2k_\perp)}\, e^{-(k_\perp+w)(\mathfrak{z}-ut)}\left[\frac{1}{\bar{\mathcal{D}}\left(\frac{(w+k_\perp)u}{\sqrt{w(w+2k_\perp)}}, -w(w+2k_\perp)\right)} - \frac{1}{\bar{\mathcal{D}}\left(-\frac{(w+k_\perp)u}{\sqrt{w(w+2k_\perp)}}, -w(w+2k_\perp)\right)}\right] , \quad z > ut \quad . \tag{74}$$

In each term of (74), the first argument of $\mathcal{D}^-$ is real, and of magnitude $\geq$ u. Since we are assuming that u >> V_σ for all σ, we may use the asymptotic forms (readily obtained from (16) for large ω/k)

$$\mathcal{D}^-\left(\pm\frac{u(w+k_\perp)}{\sqrt{w(w+2k_\perp)}}\ ,\ -w(w+2k_\perp)\right)=\mathcal{D}_1\pm i\mathcal{D}_2 \tag{75}$$

where

$$\mathcal{D}_1 \stackrel{\sim}{=} +1+\frac{\omega_o^2}{u^2(w+k_\perp)^2}\ , \tag{76}$$

$$\mathcal{D}_2=\sqrt{\pi}\sum_\sigma\frac{k_\sigma^2u(w+k_\perp)\ e^{-\frac{u^2(w+k_\perp)^2}{w(w+2k_\perp)V_\sigma^2}}}{[w(w+2k_\perp)]^{3/2}V_\sigma}\ , \tag{77}$$

where ω_o is given by (44). It follows that (74) is given approximately by

$$\phi_{1B}\stackrel{\sim}{\sim}\frac{q}{\pi^{3/2}}\int_o^{2\pi}d\phi\int_o^\infty k_\perp dk_\perp\ e^{-ik_\perp\cos\phi}$$

$$\int_o^\infty\frac{dw\ \ e^{-(k_\perp+w)(z-ut)}\sum_\sigma\frac{k_\sigma^2\ u(w+k_\perp)}{V_\sigma}\ e^{-\frac{u^2(w+k_\perp)^2}{w(w+2k_\perp)V_\sigma^2}}}{[w(w+2k_\perp)]^{5/2}\left[1+\frac{\omega_o^2}{u^2(w+k_\perp)^2}\right]^2}\ . \tag{78}$$

It is convenient to introduce a new variable by

$$w=k_\perp\left[\frac{\sqrt{v^2+1}}{v}-1\right]\ , \tag{79}$$

or

$$v=k_\perp/\sqrt{w(w+2k_\perp)}\ , \tag{80}$$

whence

$$\phi_{1B} = \frac{q}{\pi^{3/2}} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{dk_\perp\, e^{-ik_\perp R_\perp \cos\phi}}{k_\perp^2}$$

$$\int_0^{\infty} \frac{dv\; v^2 \sum_\sigma \frac{k_\sigma^2 u}{V_\sigma}\, e^{-\frac{u^2(v^2+1)}{V_\sigma^2}}}{\left[1 + \frac{\omega_o^2 v^2}{u^2 k_\perp^2 (v^2+1)}\right]^2} = \frac{q}{\pi^{3/2}} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{dk_\perp\, e^{-ik_\perp R_\perp \cos\phi}}{k_\perp^2 \left(1 + \frac{\omega_o^2}{k_\perp^2 u^2}\right)^2}$$

$$\int_0^{\infty} \frac{dv\; v^2 (v^2+1)^2}{[v^2+v_o^2(k_\perp)]^2} \sum_\sigma \frac{k_\sigma^2 u}{V_\sigma}\, e^{-\frac{u^2(v^2+1)}{V_\sigma^2}} \tag{81}$$

where

$$v_o(k_\perp) \equiv \frac{1}{\sqrt{1 + \frac{\omega_o^2}{k_\perp^2 u^2}}} \quad . \tag{82}$$

Because of the factor $e^{-\frac{u^2 v^2}{V_\sigma^2}}$, the integrand goes to zero rapidly for $v > V_\sigma/u$, therefore, an order of magnitude estimate of (81) may be obtained by replacing the factor $(1 + v^2)^2 e^{-u^2v^2/V_\sigma^2}$ by unity and the upper limit of the v integration by V_σ/u. Thus

$$\phi_{1B} \lesssim \frac{q}{\pi^{3/2}} \int_0^{2\pi} d\phi \int_0^\infty \frac{dk_\perp\, e^{ik_\perp R_\perp \cos\phi}}{k_\perp^2}$$

$$\sum_\sigma \frac{k_\sigma^2 u}{V_\sigma} e^{-u^2/V_\sigma^2} v_o^2 \int_0^{V_\sigma/u} \frac{dv\ v^2}{(v^2+v_o^2)^2} = \frac{q}{2\pi^{3/2}} \int_0^{2\pi} d\phi \int_0^\infty \frac{dk_\perp\, e^{-ik_\perp R_\perp \cos\phi}}{k_\perp^2} \sum_\sigma \frac{k_\sigma^2 u\, e^{-u^2/V_\sigma^2}}{V_\sigma}$$

$$v_o^3 \left[\arctan \frac{V_\sigma}{uv_o} - \frac{V_\sigma}{2uv_o\left(1 + \frac{V_\sigma^2}{u^2 v_o^2}\right)}\right] \tag{83}$$

In order to get an order of magnitude estimate of (83), we divide the $k_\perp$ integral into two regions, corresponding to $v_o << V_\sigma/u$ and $v_o >> V_\sigma/u$. For $V_\sigma << u$, these regions correspond approximately to $k_\perp << \omega_o V_\sigma/u^2$ and $k_\perp >> \omega_o V_\sigma/u^2$. The square bracket in (83) is easily estimated in the two regions, and assuming there is no resonance in the region $k_\perp \approx \omega_o V_\sigma/u^2$, one has approximately

$$\phi_{1B} \approx \frac{q}{2\pi^{3/2}} \int_0^{2\pi} d\phi \sum_\sigma \frac{k_\sigma^2 u\ e^{-u^2/V_\sigma^2}}{V_\sigma} \left\{ \frac{\pi}{2} \int_0^{\omega_o V_\sigma/u^2} \frac{dk_\perp\, e^{-ik_\perp R_\perp \cos\phi}}{k_\perp^2 + \frac{\omega_o^2}{u^2}} \right.$$

$$\left. \pm \frac{2V_\sigma^2}{u^2} \int_{\omega_o V_\sigma/u^2}^\infty \frac{dk_\perp\, e^{-ik_\perp R_\perp \cos\phi}}{k_\perp^2} \right\} \tag{84}$$

The order of magnitude of the remaining integrals can be estimated without difficulty, and one finds

$$\phi_{1B} \lesssim \frac{qu}{\omega_o} \sum_\sigma k_\sigma^{\,2}\, e^{-u^2/V_\sigma^{\,2}} \cong O\left(qK \frac{u}{V_{max}} \; e^{-u^2/V^2_{max}}\right) \tag{85}$$

where V_{max} is the thermal velocity of the fastest component (usually electrons). Thus for highly suprathermal particles, the branch cut contribution may be neglected.

We now turn to the calculation of the field behind the particle ($ut > \mathcal{z}$). In this case, the contour of the $k_{\mathcal{z}}$ integral in (58) must be closed in the upper half plane, and one finds, in addition to the pole and branch cut contributions, residues from the zeroes of Δ^-. We will also see that only part of the residue contributes from the pole at $k_{\mathcal{z}} = ik_\perp$. Specifically, on a small circle around $k_{\mathcal{z}} = ik_\perp$, one may write

$$k_{\mathcal{z}} = ik_\perp + \epsilon\, e^{i\gamma}, \quad \frac{-3\pi}{2} < \gamma < \pi/2 \quad , \tag{86}$$

and the corresponding dielectric function, according to (58-a) and (61) is

$$\mathcal{D}^-\left(u\sqrt{\frac{k_\perp}{2\epsilon}}\; e^{\frac{i}{2}\left(\frac{\pi}{2}-\gamma\right)},\; 2ik_\perp\,\epsilon\, e^{i\gamma}\right). \tag{87}$$

We see that the first argument of (87) is in the upper half plane. It is then easy to show from (16) that

$$\lim_{\epsilon\to o} \mathcal{D}^-\left(u\sqrt{\frac{k_\perp}{2}}\quad e^{i\left(\frac{\pi}{2}-\gamma\right)},\; 2ik_\perp\,\epsilon\, e^{i\gamma}\right) = 1 + \frac{\omega_o^{\,2}}{k_\perp^{\,2}u^2}\,, \left\{\begin{array}{l} -\frac{3\pi}{2} < \gamma < -\pi \\ 0 < \gamma < \frac{\pi}{2} \end{array}\right. ,$$

$$= \sqrt{\frac{\pi}{k_\perp\epsilon^3}}\; e^{\frac{i}{2}\left(\frac{\pi}{2}-3\gamma\right)} \sum_\sigma \frac{k_\sigma^{\,2}u}{V_\sigma} \;\exp\left[\frac{-ik_\perp u^2\, e^{-i\gamma}}{2\,V_\sigma^{\,2}}\right] = \infty,\; -\pi < \gamma < 0 \;.$$

(88)

It follows that only <u>half</u> of the residue contributes at $k_z = ik_\perp$. The result for ϕ_{1p} behind the particle is thus just one-half the result ((68) and (69) or (67) with z - ut replaced by ut $-z$) for the corresponding point in front.

The contribution from the branch cut again proves to be negligible for $u >> V_\sigma$; we now turn to the contribution from the zeroes of Δ^-. If we are very near the particle, there is no obvious reason to assume that we can neglect the contribution of any of the infinite set of zeroes of Δ^-. However, if we are <u>well behind</u> (i.e., several Debye lengths behind) the particle, we may restrict ourselves to the "least damped" poles, i.e., those with the smallest imaginary part. These are the usual Landau poles, only now expressed in terms of k_z. We assume that for these roots

$$\frac{|\mathrm{Re}\ k_z|u}{\sqrt{k_\perp{}^2+|\mathrm{Re}\ k_z|^2}} >> V_\sigma \quad , \text{ all } \sigma, \tag{89}$$

and

$$\mathrm{Im}\ k_z << \mathrm{Re}\ k_z\ . \tag{90}$$

The integral term in (16) may be approximated by

$$\int_0^L dx\, e^{x^2} = \frac{e^{L^2}}{2\ L}\left[1 + \frac{1}{2L^2} + \frac{3}{4L^4} + \cdots\right], \quad L >> 1\ , \tag{91}$$

and we may write

$$\mathcal{D}^-\left(\frac{k_z u}{\sqrt{k_z{}^2 + k_\perp{}^2}}\ , k_z{}^2 + k_\perp{}^2\right) = \mathcal{D}_1 - i\mathcal{D}_2 \tag{92}$$

where

$$\mathcal{D}_1(k_z,k_\perp) \approx 1 - \frac{\omega_o^2}{k_z^2u^2} - \frac{6\pi}{u^4}\sum_\sigma \frac{n_\sigma e_\sigma^2 V_\sigma^2(k_z^2+k_\perp^2)}{m_\sigma k_z^4}$$

$$+ \ldots , \quad k_z >> \frac{k_\perp}{\sqrt{(u^2/V_\sigma^2) - 1}} \;, \text{ all } \sigma, \tag{93}$$

and

$$\mathcal{D}_2(k_z,k_\perp) = \sqrt{\pi}\sum_\sigma \frac{k_\sigma^2 k_z u}{V_\sigma(k_z^2+k_\perp^2)^{3/2}} \exp\left[- k_z^2u^2/V_\sigma^2(k_z^2+k_\perp^2)\right] . \tag{94}$$

In view of the assumption (90), we seek zeroes of $\mathcal{D}$ of the form

$$k_z = \pm k_o + id_o, \quad |d_o| << |k_o| \tag{95}$$

and take k_o to be the positive real root of

$$\mathcal{D}_1(k_o,k_\perp) = 0 \tag{96}$$

or

$$k_o = \sqrt{\frac{\omega_o^2}{2u^2} + \frac{3\pi}{u^4}\sum_\sigma \frac{n_\sigma e_\sigma^2 V_\sigma^2}{m_\sigma} + \sqrt{\left(\frac{\omega_o^2}{2u^2} + \frac{3\pi}{u^4}\sum_\sigma \frac{n_\sigma e_\sigma^2 V_\sigma^2}{m_\sigma}\right)^2 + \frac{6\pi k_\perp^2}{u^4}\sum_\sigma \frac{n_\sigma e_\sigma^2 V_\sigma^2}{m_\sigma}}} . \tag{97}$$

It will be observed that (97) satisfies (89), provided that

$$k_\perp << \frac{\omega_o \sqrt{1 - V_{max}^2/u^2}}{V_{max}} \;, \tag{98}$$

where V_{max} is the speed of the fastest species in the system. In evaluating the $k_\perp$ integral, we will have to check for consistency; the main contribution should come from the region described by (98). So far we have assumed that $u > V_\sigma$, all σ; the stronger condition $u >> V_\sigma$ has not been needed. In view of (98), the term proportional to $k_\perp^2$ in (97) is small compared to $(\omega_o/u)^4$ and we could expand (97); however, in order to retain the maximum range of qualitative validity, it is useful to leave (97) as it stands.

To determine the damping d_o, we expand $\mathscr{D}^-$ in the usual fashion, i.e.

$$\mathscr{D}^- = \mathscr{D}_1(k_o,k_\perp) + id_o\left(\frac{\partial\mathscr{D}_1}{\partial k_z}\right)_{k_z\to k_o} + \ldots - i\mathscr{D}_2(k_o,k_\perp) + \ldots = 0 \tag{99}$$

or

$$d_o = \frac{\mathscr{D}_2(k_o,k_\perp)}{\left(\frac{\partial\mathscr{D}_1}{\partial k_z}\right)_{k_z\to k_o}} \approx \frac{\sqrt{\pi}\,u^3k_o^{\,4}\sum\limits_\sigma \frac{k_\sigma^2\, e^{-\frac{u^2k_o^{\,2}}{V_\sigma^{\,2}(k_o^{\,2}+k_\perp^{\,2})}}}{V_\sigma}}{2\omega_o^{\,2}(k_o^{\,2}+k_\perp^{\,2})^{3/2}\left[1+\frac{6\pi\sum\limits_\sigma\frac{n_\sigma e_\sigma^{\,2}V_\sigma^{\,2}}{m_\sigma}}{u^2\omega_o^{\,2}}\right]}\ . \tag{100}$$

The contribution of the Landau poles defined by (95), (97), (100) to the first term of (12) is then given by

$$\phi_{1L} = -\frac{qu^2}{\pi\omega_o^{\,2}}\int_0^{2\pi}d\phi\int_0^\infty \frac{k_\perp dk_\perp\, e^{-ik_\perp R_\perp\cos\phi}}{k_o^{\,2}(k_\perp)+k_\perp^{\,2}} \times \frac{k_o^{\,3}\, e^{-d_o(ut-z)}\ \sin k_o(ut-z)}{\left(1+\frac{6\pi}{u^2\omega_o^{\,2}}\sum\limits_\sigma\frac{n_\sigma e_\sigma^{\,2}V_\sigma^{\,2}}{m_\sigma}\right)}\ . \tag{101}$$

So far, the number and properties of the components of the plasma have been left arbitrary. For simplicity we now assume that the electrons (velocity V, mass m, charge $-e$, density n) are much lighter and much faster than any other species. Then

$$\omega_o \approx \sqrt{\frac{4\pi n e^2}{m}} \quad , \sum_\sigma \frac{n_\sigma e_\sigma{}^2 V_\sigma{}^2}{m_\sigma} \approx \frac{V^2 \omega_o{}^2}{4\pi} \tag{102}$$

We also define an electron Debye wave number by

$$k_e = \sqrt{\frac{8\pi n e^2}{m V^2}} \approx \frac{\sqrt{2}\ \omega_o}{V} \tag{103}$$

Introducing the dimensionless variable

$$x = k_\perp / k_e \quad , \tag{104}$$

and the shorthand

$$N = \omega_o (ut - z)/(\sqrt{2}u) = \left[k_e(ut - z)\right] \left(\frac{V}{2u}\right) \quad , \tag{105}$$

$$a = 1 + \frac{3V^2}{2u^2} \quad , \tag{106}$$

$$b = \frac{\omega_o (ut - z)}{\sqrt{2}\ u\ k_e R_\perp} = \frac{V(ut - z)}{2uR_\perp} \quad , \tag{107}$$

$$M = u/V \quad , \tag{108}$$

one may write (101) as*

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* Strictly speaking, the integral in (109) does not converge. This is because the root (97), (100) is only valid for $x \ll 1$. For $x \gg 1$, the appropriate root is $(k_z/k_e) \approx \frac{ix\sqrt{\log x^2}}{M}$, which gives strong damping.

$$\phi_{1L} = -\frac{1}{\pi a}(qk_e)\, M \int_0^{2\pi} d\phi \int_0^{\infty} dx\, x\, e^{-iNx\cos\phi/b}$$

$$\frac{\left(a + \sqrt{a^2 + 12x^2}\right)^{3/2} \sin N\sqrt{a + \sqrt{a^2 + 12x^2}}}{\left[a + \sqrt{a^2 + 12x^2} + 4M^2x^2\right]} \times$$

$$\times \exp\left\{ - \frac{2\sqrt{\pi}\; N M^3 \left[a + \sqrt{a^2 + 12x^2}\right]^2 e^{-M^2/\left(1 + \frac{4M^2x^2}{a + \sqrt{a^2 + 12x^2}}\right)}}{a\left[a + \sqrt{a^2 + 12x^2} + 4M^2x^2\right]^{3/2}} \right\} . \tag{109}$$

The condition (98) implies that the main contribution to (109) should come from the region

$$x \ll 1$$

In addition, we will assume in most of what follows that the point of observation is far behind the particle; specifically, a number of Debye lengths much greater than the Mach number. In other words, we assume

$$N \gg 1$$

In this limit, (109) may be evaluated approximately by the method of stationary phase. The principal theorem employed may be stated as follows: If λ is a large number and $f(y)$, $g(y)$ are continuous, infinitely differentiable functions on $[a,b]$, then

$$\int_a^b dy\; f(y)\, e^{i\lambda g(y)} \cong$$

$$\sum_{\sim}\epsilon_n f(y_n)\sqrt{\frac{2\pi}{\lambda|g''(y_n)|}}\ \exp\left\{i\ [g(y_n) + (\pi/4)\ \mathrm{sgn}(g''(y_n))]\right\} \times$$

$$\times\ [1 + 0(\lambda^{-1/2})],\ \ |\lambda g''(y_n)| >> \left|\frac{f'(y_n)}{f(y_n)}\right|^2 \quad ,$$

$$|g''(y_n)| >> |g^{(m)}(y_n)|^{2/m}\ (\lambda)^{(2/m\ -\ 1)},\ m = 3,4,\ \ldots \tag{110}$$

where the y_n are the roots (if any) on [a,b] of

$$g'(y_n) = 0 \qquad , \tag{111}$$

and

$$\begin{aligned}\epsilon_n &= (1/2),\ \text{if } y_n = a \text{ or } b,\\ &= 1 \text{ otherwise}\end{aligned} \tag{112}$$

If there are no roots of (111), the integral in (110) is $0(\lambda^{-1})$, and may be evaluated more precisely by partial integration in any given case.

Equation (109) may be cast in the form

$$\phi_{1L} = \frac{1}{2i}\int_0^{2\pi} d\phi \int_0^{\infty} dx\ A(x) \sum_{\pm} \pm\, e^{iNh_\pm(x,\phi)} \tag{113}$$

where

$$A(x) = -\frac{q\,k_e M x (a + \sqrt{a^2 + 12x^2})^{3/2}}{\pi a[a + \sqrt{a^2 + 12x^2} + 4M^2x^2]} \times$$

$$\times \exp\left\{ - \frac{2\sqrt{\pi}\, N M^3 [a + \sqrt{a^2 + 12x^2}]^2 \exp\left[- M^2 / \left(1 + \frac{4M^2x^2}{a + \sqrt{a^2 + 12x^2}}\right)\right]}{a[a + \sqrt{a^2 + 12x^2} + 4M^2x^2]^{3/2}} \right\}, \quad (114)$$

$$h_\pm(x,\phi) = \pm \sqrt{a + \sqrt{a^2 + 12x^2}} - x(\cos\phi)/b \quad (115)$$

Inasmuch as $A(x)$ is not too rapidly varying in the sense prescribed in (110), we may apply (110) to (113). One finds

$$\frac{\partial h_\pm}{\partial x} = \pm \frac{6x}{\sqrt{a(a^2 + 12x^2) + (a^2 + 12x^2)^{3/2}}} - \frac{\cos\phi}{b} \quad (116)$$

In order for real, positive roots of (116) to exist, one must have

$$\pm \cos\phi > 0 \quad (117)$$

One then finds the roots

$$x_\pm(\phi) = \left[\frac{3}{8}\left(\frac{b}{\cos\phi}\right)^4 - \frac{a}{4}\left(\frac{b}{\cos\phi}\right)^2 - \frac{a^2}{12} \pm \frac{3}{8}\left(\frac{b}{|\cos\phi|}\right)^3 \times \sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}\right]^{1/2}$$

(118)

For these roots to be real, one must require that

$$|\cos\phi| < \frac{b}{2}\sqrt{\frac{3}{a}} \quad (119)$$

The condition $x \ll 1$ cannot be satisfied by the upper sign of (118), inasmuch as one can easily show that, for that case,

$$x_+ \geq a/2 = O(1) \tag{120}$$

Therefore, we discard this root and consider only the neighborhood of

$$x_o = x_-(\phi) = \left[\frac{3}{8}\left(\frac{b}{\cos\phi}\right)^4 - \frac{a}{4}\left(\frac{b}{\cos\phi}\right)^2 - \frac{a^2}{12} - \frac{3}{8}\left(\frac{b}{|\cos\phi|}\right)^3\sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}\right]^{1/2} . \tag{121}$$

The root (121) will satisfy $x \ll 1$ if

$$|\cos\phi| \ll \frac{b}{2}\sqrt{\frac{3}{a}} \tag{122}$$

which raises the possibility of expanding (121) in powers of $\sqrt{\frac{3}{a}}\,\frac{b}{|2\cos\phi|}$. However, we will leave (121) in its present form in hopes of obtaining a qualitatively valid result for the region

$$b \gtrsim \sqrt{4a/3}$$

In order to apply (110) we need the following:

$$\left.\frac{\partial^2 h_\pm(x,\phi)}{\partial x^2}\right|_{x=x_o} = \pm\frac{|\cos\phi|^3\sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}}{4b^3x_o} \times \left[\frac{3b}{|\cos\phi|} + \sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}\right] , \tag{123}$$

and

$$h_\pm(x_o,\phi) = \pm\frac{3x_o(\cos\phi)}{ba}\left[\left(\frac{b}{\cos\phi}\right)^2 - \frac{a}{3} + \frac{b}{|\cos\phi|}\sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}\right] , \tag{124}$$

where we have used (116), (121) and the simplifying relation

$$\sqrt{a^2 + 12x_o^2} = \frac{(3/2)b}{|\cos\phi|}\left(\frac{b}{|\cos\phi|} - \sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}\right) . \tag{125}$$

It will be observed that (123) is zero for

$$\frac{b}{|\cos\phi|} = 2\sqrt{\frac{a}{3}}$$

so that the method breaks down for this region. We will return to this point later. For the present, we consider only the region

$$b > 2\sqrt{\frac{a}{3}}$$

for which $\partial^2 h_\pm/\partial x^2\Big|_{x=x_o} \neq 0$.

Then from (110), (113), (117), (121), (123), and (124) we have

$$\phi_{1L} \cong \frac{2b^{3/2}}{2i}\sqrt{\frac{2\pi}{N}}\left[\left(\int_o^{\pi/2} + \int_{3\pi/2}^{2\pi}\right)e^{iNH(\phi)}\right.$$

$$\left. - \int_{\pi/2}^{3\pi/2} e^{-iNH(\phi)}\right]\frac{d\phi}{|\cos\phi|}\; A(x_o)\sqrt{\frac{x_o(\phi)}{\sqrt{\frac{b^2}{\cos^2\phi} - \frac{4a}{3}}\left[3b + \sqrt{\frac{b^2}{\cos^2\phi} - \frac{4a}{3}}\right]}}\;,$$

$$b - 2\sqrt{a/3} >> N^{-2/3} \qquad (126)$$

where A(x) is given in (114), and where

$$H(\phi) = \frac{3x_o(\phi)\,|\cos\phi|}{ab}\left[\left(\frac{b}{\cos\phi}\right)^2 - \frac{a}{3} + \frac{b}{|\cos\phi|}\sqrt{\left(\frac{b}{\cos\phi}\right)^2 - \frac{4a}{3}}\right] + \frac{\pi}{4}\;. \qquad (127)$$

The ϕ integral may also be done by the method of stationary phase. It is evident that

$$H'(\phi) = \pm\left[\frac{\partial h_\pm(x,\phi)}{\partial\phi}\Bigg|_{x=x_o(\phi)} + \frac{\partial x_o(\phi)}{\partial\phi}\,\frac{\partial h(x,\phi)}{\partial x}\Bigg|_{x=x_o(\phi)}\right] = \pm\, x_o(\phi)(\sin\phi)/b\;. \qquad (128)$$

But $x_o(\phi)$ is not zero unless $\cos\phi = 0$, in which case the coefficient of $e^{\pm iNH(\phi)}$ in (126) is zero; therefore, the contribution from the region around $x_o(\phi) = \cos\phi = 0$ is negligible. Thus the roots of interest are those for which

$$\sin\phi_n = 0$$

or

$$\phi_n = (0,\ \pi,\ 2\pi) \tag{129}$$

It follows that

$$H''(\phi_n) = \pm\, x_o(\phi_n)(\cos\phi_n)/b = x_o(\phi_n)/b \quad , \tag{130}$$

where we have used (117). The theorem (110) may now be applied to (126) to obtain

$$\phi_{1L} \cong \frac{4\pi b^2 A[x_o(\phi_n)]\ \cos N[H(\phi_n) - \frac{\pi}{4}]}{N\sqrt{b^2 - \frac{4a}{3} + 3b\sqrt{b^2 - \frac{4a}{3}}}}\ ,\ b - \sqrt{4a/3} >> N^{-2/3} \tag{131}$$

(it will be noted that $x_o(\phi_o) = x_o(\phi_1) = x_o(\phi_2)$).

Equations (131), (129), (127), (121) and (114) give a complete description of the "Landau" contribution to the potential for the region

$$b - 2\sqrt{a/3} >> N^{-2/3}$$

(this restriction is imposed because of the restriction $|g''| >> |(g''')^2/\lambda|^{1/3}$ in (110)). The result (131) may be simplified considerably except in the neighborhood

$$b - \sqrt{4a/3} \approx 1$$

For $b >> \sqrt{4a/3}$, everything may be expanded in powers of $\sqrt{4a/3b^2}$. One has

$$x_o(\phi_n) = (a^{3/2}/3\sqrt{2}\ b)\left[1 + \frac{5a}{12b^2} + 0\left(\frac{a^2}{b^4}\right)\right] \quad ,$$
$$b >> \sqrt{4a/3} \tag{132}$$

and

$$H(\phi_n) - \pi/4 = \sqrt{2a}\left[1 - \frac{a}{12b^2} + 0\left(\frac{a^2}{b^4}\right)\right] \quad , \quad b^2 >> 4a/3. \tag{133}$$

Using also (109), one finds

$$\phi_{1L} = -\frac{2qk_eMa\left[1 + \frac{13a}{12b^2} + 0\left(\frac{a^2}{b^4}\right)\right]}{3N\left[1 + \frac{a}{6b^2} + \frac{M^2a^2}{9b^2}\right]} \quad \times$$

$$\times \cos\left\{N\sqrt{2a}\ [1 - a/12b^2 + 0(a^2/b^4)]\right\} \exp\left\{-\frac{2N\sqrt{2\pi/a}\ e^{-1/[M^{-2}+a^2/9b^2]}}{[M^{-2} + a^2/9b^2]^{3/2}}\right\}$$
$$b >> \sqrt{4a/3} \tag{134}$$

Similarly, if

$$N^{-2/3} << b - \sqrt{4a/3} << 1 \ ,$$

one may ignore terms $\sqrt{b^2 - 4a/3}$ compared to unity, which gives

$$x_o(\phi_n) = \frac{a}{2}\left[1 - \frac{4\sqrt{b - \sqrt{4a/3}}}{a^{1/4}3^{3/4}} + \frac{10(b - \sqrt{4a/3})}{3^{1/2}\sqrt{a}} + 0[(b - \sqrt{4a/3})^{3/2}]\right]$$
$$b - \sqrt{4a/3} << 1 \quad , \tag{135}$$

$$H(\phi_n) - \frac{\pi}{4} = \frac{3\sqrt{3a}}{4}\left[1 + \frac{b - \sqrt{4a/3}}{2\sqrt{3a}} + 0\left((b - \sqrt{4a/3})^{3/2}\right)\right] , \tag{136}$$

and

$$\phi_{1L} = -\frac{4qk_e\cdot 3^{3/8}a^{1/8}\exp\left[-\frac{18\sqrt{\pi}Ne^{-3/a}}{a^2}\right]\cos\left[\frac{3\sqrt{3a}N}{4}\left(1 + \frac{b-\sqrt{4a/3}}{2\sqrt{3a}}\right)\right]}{M\,N\left(1 + \frac{3}{m^2a}\right)(b - \sqrt{4a/3})^{1/4}}\left[1 + 0(N^{-1/2})\right.$$

$$\left. + 0(\sqrt{b^2 - 4a/3})\right] , \quad N^{-2/3} << b - \sqrt{4a/3} << 1 \quad . \tag{137}$$

Inasmuch as "a" is 0(1), we see that the potential is exponentially damped with a damping length of order (Debye length $\times$ Mach number).

We now turn to the region

$$b - \sqrt{4a/3} << N^{-2/3} \qquad ,$$

where our previous approach breaks down. The reason for this breakdown is very simple; for this region, not only $h'_\pm(x_o)$ but also $h''_\pm(x_o)$ is nearly zero. Thus we have a higher order stationary phase point at $x = x_o$. This situation may be treated by standard techniques. One needs the third derivative

$$\left.\frac{\partial^3 h_\pm(x,\phi)}{\partial x^3}\right|_{\substack{x=x_o(\phi_n)\\ \phi=\phi_n}} \approx \mp\frac{1}{8}\left(\frac{3}{a}\right)^{5/2} , \quad b - \sqrt{4a/3} << 1. \tag{138}$$

In this limit it is not difficult to deduce that

$$\int_0^\infty dx\, A(x)\, e^{iNh_\pm(x,\phi)} \approx A(x_o)\, e^{iNh_\pm(x_o,\phi)}\int_{-x_o}^\infty d(x-x_o)\, e^{\frac{iN\partial^3 h_\pm(x,\phi)/\partial x^3\big|_{x=x_o}(x-x_o)^3}{6}}$$

$$= \left(\frac{6}{N \left| \frac{\partial^3 h_\pm(x,\phi)}{\partial x^3} \right|_{x=x_o}} \right)^{1/3} A(x_o)\, e^{iNh_\pm(x_o,\phi)} \int_{-\left(\frac{N\left|\partial^3 h_\pm/\partial x^3\right|_{x=x_o}}{6}\right)^{1/3} x_o}^{\infty} dz\, e^{i\, \mathrm{sgn}(\partial^3 h_\pm/\partial x^3) z^3}$$

$$\simeq \left| 6 \Big/ N \frac{\partial^3 h_\pm(x,\phi)}{\partial x^3}_{x=x_o} \right|^{1/3} A(x_o)\, e^{iNh_\pm(x_o,\phi)} \int_{-\infty}^{\infty} dz\, e^{iz^3 \mathrm{sgn}\, h'''(x_o,\phi)}$$

$$= \left| \frac{6}{Nh'''_\pm(x_o(\phi),\phi)} \right|^{1/3} A(x_o)\, e^{iNh_\pm(x_o,\phi)} \sqrt{3} \left(\frac{1}{3}!\right) \tag{139}$$

The ϕ integral can be done in exactly the same manner as before, and one finds

$$\phi_{1L} = - \frac{qk_e 2^{11/6}\, 3^{5/4}\, a^{1/12} \left(\frac{1}{3}\right)!}{MN^{5/6} \sqrt{\pi}\, (1 + 3/M^2 a)} \exp\left[- \frac{18\sqrt{\pi} N e^{-3/a}}{a^2} \right]$$

$$\times \cos\left[\frac{3\sqrt{3a}N}{4} \left(1 + \frac{b - \sqrt{4a/3}}{2\sqrt{3a}} \right) - \frac{\pi}{4} \right] ,$$

$$b - \sqrt{4a/3} \ll N^{-2/3} \tag{140}$$

We now turn to the region well outside the Mach cone, i.e.

$$b \ll \sqrt{4a/3} \qquad ,$$

where the moment equations predict no induced fields. In this region, the factor $e^{-iNx(\cos\phi)/b}$ is dominant, and the main contribution comes from the region

$$\frac{b}{N} \lesssim x << 1 \quad .$$

Noting that

$$\int_0^{2\pi} d\phi\, e^{-iNx(\cos\phi)/b} = 2\pi J_0(nx/b) \ ,$$

one estimates from (109)

$$\phi_{1L} \approx -\,2qk_e \sqrt{2/a}\, M \sin\sqrt{2a}\, N \int_0^{x_0} dx\, xJ_0(Nx/b)$$

$$= -\,2qk_e \sqrt{2/a}\, M\left(\frac{b}{N}\right) x_0 J_1(Nx_0/b) \sin\sqrt{2a}\, N \quad ,$$

$$b << 1 \quad ,$$

where x_0 is chosen to be much less than unity, but greater than b/N. A good estimate should be obtained by taking $x_0 = 1$ and using the asymptotic form for the Bessel function, which leads to

$$\phi_{1L} \approx -\frac{4qk_e M}{\sqrt{\pi a}}\left(\frac{b}{N}\right)^{3/2} \sin\sqrt{2a}\, N \sin\left(\frac{N}{b} - \frac{\pi}{4}\right) \quad , \tag{141}$$

$$N/b >> 1$$

The same estimate is also valid for $N < 1$, provided that

$$N/b \equiv k_e R_\perp << 1.$$

This leaves unestimated (aside from the "transition regions" $R_\perp \approx k_e^{-1}$, $ut - z \approx u/\omega_0$) only the interior of the cylinder defined by

$$k_e R_\perp << 1, \quad \omega_0(ut - z)/u << 1.$$

For this region, it is convenient to rewrite the first term of (12) in spherical coordinates and subtract out the Coulomb field, i.e.

$$\phi_{1i} \equiv \phi_1 - \frac{q}{R} = \frac{q}{2\pi^2}\int_0^\infty dk \int_{-1}^{1} d\mu \int_0^{2\pi} d\phi$$

$$e^{ik[(ut-z)\mu+R_\perp\sqrt{1-\mu^2}\cos\phi]}\left[\frac{1}{\Delta^-(ku\mu,k)} - 1\right]$$

$$= -\frac{qk_e^2}{2\pi^2}\int_0^\infty dk \int_{-1}^{1} d\mu \int_0^{2\pi} d\phi\, e^{ik[(ut-z)\mu+R_\perp\sqrt{1-\mu^2}\cos\phi]} \frac{Y(M\mu)}{k^2 + k_e^2 Y(M\mu)}$$

(142)

where

$$Y(x) \equiv 1 - xe^{-x^2}\left[\sqrt{\pi}\, i + 2\int_0^x dv\, e^{v^2}\right] \tag{143}$$

Introducing the notation

$$\delta(\mu,\phi) \equiv k_e\,[(ut - z)\mu + R_\perp\sqrt{1-\mu^2}\cos\phi] \tag{144}$$

and the substitution

$$k = k_e v/\delta(\mu,\phi) \quad , \tag{145}$$

one may cast (142) in the form

$$\phi_{1i} = -\frac{qk_e}{2\pi^2}\int_0^{2\pi} d\phi \int_{-1}^{1} d\mu \int_0^{\infty\,\mathrm{sgn}\,\delta(\mu,\phi)} dv\, \frac{\delta(\mu,\phi)\; Y(M\mu)\, e^{iv}}{v^2 + \delta^2\; Y(M\mu)} \quad . \tag{146}$$

Because of the Y in the numerator, the main contribution to the μ integration will come from $\mu \lesssim 1/M$; from the definition (144) and our assumptions about the smallness of $k_e R_\perp$, $k_e(ut - \mathfrak{z})/M$, we then have

$$|\delta(\mu,\phi)| << 1, \quad \text{all } \mu,\phi \quad . \tag{147}$$

From the definition (143), it is evident that

$$|Y(x)| \lesssim 1, \quad \text{all } x, \tag{148}$$

so the second term in the denominator of the integrand in (146) is a small (complex) quantity.

The v integration is not difficult to do in terms of known integrals, although one must be careful with the phases. One finds, using partial fractions,

$$\phi_{1i} = -\frac{qk_e}{2\pi^2}\int_o^{2\pi} d\phi \int_{-1}^{1} d\mu Y^{1/2}(M\mu)\,\mathrm{sgn}(\delta(\mu,\phi))$$

$$\left\{\frac{e^{+|\delta|Y^{1/2}}}{2}\int_o^{\alpha_o} d\alpha\, e^{-|\delta Y^{1/2}|e^{i\alpha}} - e^{-|\delta|Y^{1/2}}\left[\frac{1}{2}\int_o^{\alpha_o+\pi} d\alpha\, e^{-|\delta Y^{1/2}|e^{i\alpha}} - \pi S(+\delta)\right]\right.$$

$$\left.+ i \sinh |\delta|Y^{1/2}\int_{|\delta Y^{1/2}|}^{\infty}\frac{du\ e^{-u}}{u}\right\} = -\frac{qk_e}{2\pi^2}\int_o^{2\pi} d\phi \int_{-1}^{1} d\mu \left\{+\frac{\pi}{2}Y^{1/2}(M\mu)\right.$$

$$\left.+ Y(M\mu)\left[i\delta(1-\gamma-\log|\delta Y|-\frac{i\ \arg Y}{2}) - \frac{\pi}{2}|\delta|\right] + O(\delta^2)\right\} \quad . \tag{149}$$

Here $S(\delta)$ is a step function, and we have taken

$$-\pi < \arg Y(M\mu) < \pi \quad , \tag{150}$$

so that $Y^{1/2}$ is in the right half plane, and used the definition

$$\alpha_o = \frac{1}{2} \arg Y(M\mu) \tag{151}$$

The ϕ integrations may be done with the aid of the known integrals (see Appendix)

$$\int_o^{2\pi} d\phi \; |a + b \cos \phi| = 2\pi |a| \; S(|a| - b)$$

$$+4[a \arcsin \frac{a}{b} + \sqrt{b^2 - a^2}] \; S(b - |a|) \quad , \tag{152}$$

and

$$\int_o^{2\pi} d\phi (a + b \cos \phi) \log |a + b \cos \phi|$$

$$= 2\pi a \Bigg\{ 1 - \log 2 + S(b - |a|) \log b$$

$$+ S(|a| - b) \left[\log(|a| + \sqrt{a^2 - b^2}) - \frac{\sqrt{a^2 - b^2}}{|a|} \right] \Bigg\} \quad , \tag{153}$$

where S is again a step function.

It follows that

$$\phi_{1i} = - \frac{q\omega_o}{u} \Bigg\{ C_1 + \frac{\omega_o (ut-\mathfrak{z})}{u} \left[- C_2 \left(\gamma + \log \frac{\omega_o (ut-\mathfrak{z})}{\sqrt{2}\, u} \right) \right.$$

$$\left. C_3 + C_4 + C_5(\theta) \right] + \frac{C_6(\theta) \omega_o |\underset{\sim}{u}t - \underset{\sim}{R}|}{u} \Bigg\} \quad , \tag{154}$$

where

$$C_1 = \frac{1}{\sqrt{2}} \int_{-M}^{M} dw \; Y^{1/2} (w) \quad , \tag{155}$$

$$C_2 = \frac{2i}{\pi} \int_{-M}^{M} dw \; w \; Y(w) \quad , \tag{156}$$

$$C_3 = - \int_{-M}^{M} dw \; |w| Y(w) \quad , \tag{157}$$

$$C_4 = \frac{-i}{\pi} \int_{-M}^{M} dw\ wY(w) \left[\log|Y(w)| + i\ \arg Y(w)\right] , \tag{158}$$

$$C_5(\theta) = \frac{-2i}{\pi} \int_{-M}^{M} dw\ wY(w) \left\{ S(M\sin\theta - |w|) \left[\log \sqrt{M^2 - u^2}\ \tan\theta \right.\right.$$

$$\left.\left. + i\,\mathrm{sgn}\ w\ \cos^{-1}\left(\frac{|w|\cot\theta}{\sqrt{M^2 - w^2}}\right)\right] + S(|w| - M\sin\theta)\ \log\left(|w| + \frac{\sqrt{w^2 - M^2\sin^2\theta}}{\cos\theta}\right)\right\} , \tag{159}$$

$$C_6(\theta) = \frac{-2}{\pi} \int_{-M}^{M} dw\ Y(w) \left[\sqrt{M^2\sin^2\theta - w^2}\quad S(M\sin\theta - |w|)\right.$$

$$\left. - i\ \mathrm{sgn}\ w\ \sqrt{w^2 - M^2\sin^2\theta}\ S(|w| - M\sin\theta)\right] . \tag{160}$$

Here θ is the angle between $(\underset{\sim}{u}t - \underset{\sim}{R})$ and $\underset{\sim}{u}$. We show in the Appendix how the integrals (155)-(160) may be evaluated approximately. The result is

$$\phi_{1i} = \frac{q\omega_o}{u}\left\{\frac{\pi}{2} + \frac{\omega_o(ut-z)}{u}\left[-\gamma - 2\log\frac{\omega_o R_\perp}{u} + \log\left\{\frac{\omega_o}{u}(ut - z + |\underset{\sim}{R} - \underset{\sim}{u}t|)\right\}\right.\right.$$

$$\left.\left. + A(\theta)\right] + \frac{\omega_o|\underset{\sim}{u}t - \underset{\sim}{R}|}{u}\left[-1 + B(\theta)\right]\right\} , \tag{161}$$

where

$$A(\theta) = -\frac{4}{\sqrt{\pi}} \int_{M\sin\theta}^{M} dw\ w^7 e^{-w^2} \log\left(\frac{w\cos\theta + \sqrt{w^2 - M^2\sin^2\theta}}{w\cos\theta - \sqrt{w^2 - M^2\sin^2\theta}}\right) \tag{162}$$

and

$$B(\theta) = \frac{8}{\sqrt{\pi}} \int_{M\sin\theta}^{\infty} dw\ we^{-w^2} \sqrt{w^2 - M^2\sin^2\theta} \tag{163}$$

The functions, A, B have the following properties:

$$A(\theta) \cong \log \frac{M^2\sin^2\theta}{4} - \frac{8}{\sqrt{\pi}} \int_0^{\infty} dw\ w^7 e^{-w^2} \log w, \quad M\sin\theta \ll 1$$

$$\approx -2\ e^{-M^2\sin^2\theta} \quad , \ M\sin\theta \gg 1 \qquad ; \tag{164}$$

$$B(\theta) \cong 2 \quad , \quad M\sin\theta \ll 1$$

$$\approx 2\ e^{-M^2\sin^2\theta} \quad , \quad M\sin\theta \gg 1 \tag{165}$$

In particular, we note two limits: (1) At the midplane, $ut - z = 0$, and

$$\phi_{1i}\Big|_{ut-z=0} = -\frac{q\omega_o}{u} \left\{ \frac{\pi}{2} - \frac{\omega_o R_\perp}{u} \right\} \tag{166}$$

Comparison with (69) shows that the potential is continuous across the midplane. (2) On the axis, $R_\perp = 0 = \sin\theta$,

$$\phi_{1i}\Big|_{R_\perp=0} = -\frac{q\omega_o}{u} \left\{ \frac{\pi}{2} - \frac{\omega_o(ut-z)}{u} \left[\gamma + \log \frac{2\omega_o(ut-z)}{u} - \log M^2 - 1 \right.\right.$$

$$\left.\left. + \frac{8}{\sqrt{\pi}} \int_0^{\infty} dw\ w^7 (\log w) e^{-w^2} \right] \right\} \tag{167}$$

This is also continuous at the particle, but the corresponding electric field is both discontinuous and singular, just as in the moment equation treatment.

Finally, we should say a word about the "transients" given by the second term of (12). This integral may be written approximately in cylindrical coordinates as

$$\phi_2 \simeq \frac{q\omega_o}{4\pi^2} \int_0^{2\pi} d\phi \int_0^{\infty} dk_\perp k_\perp \, e^{-ik_\perp R_\perp \cos\phi} \int_{-\infty}^{\infty} \frac{dk_\parallel \, e^{-ik_\parallel z}}{k_\parallel^2 + k_\perp^2} \lim_{\to 0} \left[\frac{e^{i\omega_o t}}{\omega_o - k_\parallel u + i\epsilon} + \frac{e^{-i\omega_o t}}{\omega_o + k_\parallel u - i\epsilon} \right] , \qquad 168)$$

where we have used the facts that

$$\mathrm{Re}\,\omega_\pm \simeq \pm\, \omega_o, \quad \mathrm{Im}\,\omega_\pm << \mathrm{Re}\,\omega_\pm \quad ,$$

$$\left. \frac{\partial \Delta^-(\omega,k)}{\partial \omega} \right|_{\omega=\omega_\pm(k)} \simeq \pm \frac{2}{\omega_o} \qquad (169)$$

The use of (169) implies that the main contribution comes from small k, which should be checked in the course of evaluating the integral (the contribution from large k will, of course, be strongly damped). We naturally assume that the point of observation is in front of the point where the particle is injected, i.e. $z > 0$ (in a more realistic treatment we would assume that the half space $z < 0$ was empty). The $k_\parallel$ integration may be done immediately and one finds

$$\phi_2 = \frac{q\omega_o}{4\pi}\int_0^{2\pi} d\phi \int_0^{\infty} dk_\perp\, e^{-k_\perp[z+iR_\perp\cos\phi]}\left[\frac{e^{i\omega_o t}}{\omega_o+ik_\perp u}+\frac{e^{-i\omega_o t}}{\omega_o-ik_\perp u}\right]$$

$$= \frac{q\omega_o}{4\pi i u}\int_0^{2\pi} d\phi \left\{ e^{i\omega_o\left(t-\frac{\ell(\phi)}{u}\right)}\int_{-\frac{i\omega_o}{u}}^{\infty} dy\, \frac{e^{-y\ell(\phi)}}{y} - e^{-\frac{i\omega_o(ut-\ell(\phi))}{u}}\int_{\frac{i\omega_o}{u}}^{\infty}\frac{dy\, e^{-y\ell(\phi)}}{y}\right\}, \tag{170}$$

where

$$\ell(\phi) \equiv z + iR_\perp\cos\phi \tag{171}$$

The y integrals in (170) are more or less standard, and we will do them approximately in two limits. First, for $\omega_o R/u \gg 1$, the y integrations may be done by partial integration, and we find

$$\phi_2 \cong \frac{q\cos\omega_o t}{2\pi}\int_0^{2\pi}\frac{d\phi}{\ell(\phi)} = \frac{q\cos\omega_o t}{R} \tag{172}$$

where we have used (71). On the other hand, for $\omega_o R/u \ll 1$, we use

$$\int_\epsilon^\infty \frac{dx\, e^{-x}}{x} \sim -\gamma - \log\epsilon \tag{173}$$

to obtain

$$\phi_2 \cong \frac{q\omega_o}{2\pi u}\int_0^{2\pi} d\phi \left\{\pi\cos\omega_o t - \sin\omega_o t\left[\gamma + \log\frac{\omega_o\ell(\phi)}{u}\right]\right\}$$

$$= \frac{q\omega_o}{u}\left\{\pi\cos\omega_o t - \sin\omega_o t\left[\gamma + \log\left[\frac{\omega_o(z+R)}{2u}\right]\right]\right\} \tag{174}$$

Thus it appears that there is a residual oscillating potential around the initial portion of the injected particle. The time average of this potential is zero, and there is no charge density associated with it, as can be seen by calculating

$$\rho_2 = -\frac{\nabla^2\phi_2}{4\pi} \tag{175}$$

from (172), (174) or more generally from (168). (There is, however, some induced charge behind the initial position.) This result differs from the moment equation treatment of I, at least in part because of a different way of dividing the potential and charge density into "steady-state" and "transient" parts. However, for the transient contribution to the drag force we find

$$F_T = -q\left.\frac{\partial\phi_2}{\partial z}\right|_{\substack{R_\perp=0\\ z=ut}} \underset{\sim}{\simeq} \frac{q^2\cos\omega_o t}{u^2t^2}\;, \qquad \omega_o t \gg 1 \tag{176}$$

in agreement with I (the distinction between transient and steady-state would appear to be rather meaningless for $\omega_o t \lesssim 1$). For fixed $|\underset{\sim}{R}-\underset{\sim}{u}t|$, it will be noted from (172) that the transient electric field is proportional to $(\omega_o t)^{-2}$ as $\omega_o t \to \infty$.

Before comparing our results with the moment equation treatment, we make a couple of remarks about the charge density. First, the induced charge density in front of the particle is zero, in agreement with I. This can be shown by calculating

$$\rho_i = -\frac{1}{4\pi}\nabla^2\phi_i \tag{177}$$

from the approximate expressions (68), (79), or more generally from (67). Secondly, the total induced charge behind the particle exactly cancels the charge of the test particle, if transients are ignored. This can be shown by calculating $\oint \underset{\sim}{E}\cdot d\underset{\sim}{s}$ where the integral is over the surface of a large sphere. Since ϕ_{1p} and ϕ_{1L} both die out faster than R^{-1} at ∞, the integral gives zero. However, one should note that the surface must be a very large one indeed. According to (134) and (105)-(108), the potential on the axis ($b \to \infty$) is given by

$$\phi_{1L} \cong -\frac{4qM^2}{3(ut-\mathfrak{z})} \cos\left[\frac{\omega_o(ut-\mathfrak{z})}{u}\right] \times \exp\left\{-\sqrt{2\pi}\, k_e(ut-\mathfrak{z})M^2\, e^{-M^2}\right\} , \quad (178)$$

so that exponential decay does not set in until

$$k_e(ut-\mathfrak{z}) \approx M^{-2}\, e^{M^2} ,$$

which can be a very large number for large M (for example, if $M = 10$, $M^{-2} e^{M^2} \cong 10^{43}$!).

A more complete treatment which included correlations or "collisions" would presumably show that the "wake" undergoes "correlation damping" and thus extends only for about a mean free path, even well inside the cone, and even for very large Mach numbers. As noted previously (cf. (137), (138)) the field near the cone is damped out in a relatively short distance. Strictly speaking, the integral for the total charge from the moment equation treatment diverges; however, if a small phenomenological collision term is added, one reaches the same conclusion about zero total charge.

The regions where we have calculated the potential are depicted in Fig. 4; we propose to make a step by step comparison with I. In order to make such a comparison, we note the slight difference in definition of the thermal velocity and Debye length; v_o and k_o of I are related to V by

$$2v_o^2 = 3V^2, \; k_D^2 \equiv \omega_o^2/v_o^2 = 2\omega_o^2/3V^2 \tag{179}$$

We have

(1) In front of the particle (Regions I and II)

The relevant equations are (68) and (69), indicating an induced potential which screens the Coulomb potential at large distances (giving an R^{-3} dependence for the total potential) but which is much smaller than the Coulomb potential for small distances. As noted previously, the induced charge density is zero. These results are in quantitative agreement with I, as may be seen from equations (135), (29) of the latter (the potential was not calculated as accurately [cf. (I.A_7-3), (I.A_7-8), (I.A_7-4) and (I.A_7-6)] but this is not a defect of the moment equation approach).

(2) Well behind the particle and well inside the Mach cone (Region III)

Since only the charge density was computed in I, we straightforwardly calculate the latter by applying (177) to (134); using also (103) and (106)-(108), one finds*

$$\rho_i = -\frac{qk_e^2}{6\pi(ut-\mathfrak{z})}\left[1+\frac{3}{M^2}+\frac{13M^2R_\perp^2}{3(ut-\mathfrak{z})^2}+0\left(M^{-4},\frac{R_\perp^2}{(ut-\mathfrak{z})^2},\frac{u}{\omega_o|\underset{\sim}{R}-\underset{\sim}{u}t|},\frac{M^4R_\perp^4}{(ut-\mathfrak{z})4}\right)\right]$$

$$\cos\left\{\frac{\omega_o(ut-\mathfrak{z})}{u}\left[1+\frac{3}{4M^2}-\frac{M^2R_\perp^2}{3(ut-\mathfrak{z})^2}+0\left(M^{-4},\frac{R_\perp^2}{(ut-\mathfrak{z})^2},\frac{M^4R_\perp^4}{(ut-\mathfrak{z})^2}\right)\right]\right\}, \tag{180}$$

* As in Regions I and II, the "pole" contribution (67) or (68), (69), gives zero charge density.

where we have dropped the exponential factor in (134), thus assuming

$$k_e(ut-\mathfrak{z}) << M^{-2}e^{M^2} \tag{181}$$

To the same approximation, one may write equation (29) of I as

$$\rho_i \approx -\frac{qk_e^2}{6\pi(ut-\mathfrak{z})}\left[1+\frac{M^2R_\perp^2}{3(ut-\mathfrak{z})^2}\right]\cos\left\{\frac{\omega_o(ut-\mathfrak{z})}{u}\left[1+\frac{3}{4M^2} - \frac{R_\perp^2M^2}{3(ut-\mathfrak{z})^2}\right]\right\} \tag{I.29}$$

We see that the oscillating factor in (180), (I.29) is in exact agreement to the order calculated, and the coefficient agrees to leading order, though there is some discrepancy in the corrections. Thus the agreement in Region III is very good indeed.

(3) On the cone (Region IV)

The neighborhood of the Mach cone is the region which produces the greatest discrepancy between our results and those of the moment equation treatment. The relevant equation is (140), and one readily shows that

$$\rho_i \cong -\frac{qk_e^2(\omega_o/u)^{1/6}\,3^{5/4}\left(\frac{1}{3}\right)!\,\exp\left[-\frac{9\sqrt{2\pi}\;\omega_o(ut-\mathfrak{z})e^{-3}}{u}\right]}{\sqrt{2}\;\pi^{3/2}(ut-\mathfrak{z})^{5/6}}\times$$

$$\times\cos\frac{3^{3/2}\omega_o(ut-\mathfrak{z})}{2^{5/2}u}\;,\quad R_\perp = \frac{\sqrt{3}\;(ut-\mathfrak{z})}{4M} \tag{182}$$

The corresponding result* from I is

---

* Note that our Mach cone is narrower by a factor of $2\sqrt{2}$ than that of I.

$$\rho_i = \infty \qquad R_\perp = \sqrt{\frac{3}{2}}\ \frac{(ut-z)}{M} \tag{I.29}$$

Thus we have obtained the strikingly different result that the charge density is not only finite on the cone, but is damped* out in a distance of order M/k_e. The damping is, of course, due to the imaginary part of the dielectric function, which is, of course, missed by any moment equation approach. The fact that the charge density is finite, however, seems to be due to a more accurate calculation of the real part.

(4) Outside the cone (Regions V and VI)

The relevant equations are (141), (68), (69). The regions V and VI differ only in the form of the potential, not the charge density, because only the "pole" contribution, which gives zero charge density, has a different form in the two regions. From (141) the charge density dies out as

$$(\text{oscillating factor})/(k_e R_\perp)^{3/2} \quad ,$$

whereas the moment equations predict that it is zero there. While the decay of the charge density outside the cone is rather slow, the neglect of this charge is probably adequate for most purposes.

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* It should be pointed out that we have neglected some terms which are not exponentially damped and therefore dominate for $ut-z > M/k_e$. One class of such terms is described by (141) for $k_e R_\perp \gg 1$.

(5) Near and behind the particle (Regions VII and VIII)

The appropriate equations are (161)-(163). In calculating the induced charge density via (177), one should differentiate (162), (163) directly, using integration by parts when convenient, rather than using the approximate forms (164), (165). The latter are adequate for calculating the potential (and, to a certain extent, the field) but not the charge density, which involves second derivatives. One finds

(a) Well outside the cone (Region VII)

The charge density may be shown to die off in a Gaussian fashion, i.e.

$$\rho_i \approx \frac{2q\,\omega_o^{\,2}\,M^2\sin^2\theta\, e^{-M^2\sin^2\theta}}{\pi V^2\,|\underset{\sim}{R}-\underset{\sim}{u}t|}\,, \quad M\sin\theta \gg 1 \tag{183}$$

This is in good agreement with the moment equations which predict no charge outside the Mach cone.

(b) Well inside the cone (Region VIII)

$$\rho_i \approx -\frac{q\omega_o^{\,2}}{\pi V^2(ut-z)}\,, \qquad M\sin\theta \ll 1 \tag{184}$$

On the other hand, the moment equations give [cf. (I.29)]

$$\rho_i \approx -\frac{q\omega_o^{\,2}}{3\pi V^2(ut-z)}\,, \tag{185}$$

i.e. smaller by a factor of 1/3, similar to the error in the Debye length. We should be happy with such qualitative agreement in a region where there is no reason to expect much accuracy from the moment equations. While it is difficult to calculate the charge density precisely at the Mach cone, it is abundantly clear from (161)-(163) that there is no singularity there (except at the particle). Finally, one should note that the Vlasov equation breaks down very near the particle (at a radius of about the Landau length $\lambda_L \approx e^2/mV^2$).

To summarize, we have found that the moment equations have a much greater validity for fast particles than for slow particles except in the neighborhood of the Mach cone. We should also remark that our results showing a somewhat "fuzzy" Mach cone are qualitatively similar to those obtained for a finite "blob" of charge in I.

APPENDIX: EVALUATION OF SOME ANGULAR INTEGRALS OCCURRING IN (149)

In this appendix we evaluate the integrals necessary to get from equation (149) of Section 4 to (161). We begin by proving (152), (153). Let "a" be a real number and "b" a real positive number. Then it is evident that

$$\int_0^{2\pi} d\phi \; |a + b \cos \phi| = |a| \int_0^{2\pi} d\phi \; (1 + \frac{b}{a} \cos\phi) = 2\pi |a|, \; b < |a| \; . \quad \text{(A1)}$$

For $b > |a|$, we define

$$\phi_1 = \text{arc cos } (-a/b), \quad 0 \leq \phi_1 \leq \pi \quad . \quad \text{(A2)}$$

Then it is evident that $a + b \cos\phi$ will be negative for $\phi_1 < \phi < 2\pi - \phi_1$ and positive for the rest of the range. Thus

$$\int_0^{2\pi} d\phi \; |a + b \cos\phi| = \left[\int_0^{\phi_1} + \int_{2\pi-\phi_1}^{2\pi} - \int_{\phi_1}^{2\pi-\phi_1} \right] d\phi (a + b \cos\phi)$$

$$= - a(2\pi - 4\phi_1) + 4b \sin \phi_1 = -4a\left(\frac{\pi}{2} - \text{arc cos}(-a/b)\right) + 4\sqrt{b^2 - a^2}$$

$$= 2\pi \; a \; \text{arc sin}(a/b) + 4\sqrt{b^2 - a^2} \quad , \quad b > |a| \; , \quad \text{(A3)}$$

which completes the proof of (152). For the integral in (153), if $|a| > b$, we may expand the logarithm, and only even powers of $\cos\phi$ will give a contribution. One finds

$$\int_{o}^{2\pi} d\phi \; (a + b\cos\phi) \; \log \; |a + b\cos\phi| = \int_{o}^{2\pi} d\phi \; (a + b\cos\phi) \left[\log \; |a| \right.$$

$$\left. + \log \left(1 + \frac{b}{a}\cos\phi\right)\right] = 2\pi a \; \log \; |a| - a \int_{o}^{2\pi} d\phi \sum_{p=1}^{\infty} \left(\frac{b}{a}\right)^{2p} \frac{(\cos\;)^{2p}}{2p}$$

$$+ b\int_{o}^{2} d\phi \sum_{p=1}^{\infty} \left(\frac{b}{a}\right)^{2p-1} \frac{(\cos\phi)^{2p}}{2p-1} = 2\pi a \left[\log|a| + \sum_{p=1}^{\infty} \frac{(2p)!}{(p!)^2}\left(\frac{b}{2a}\right)^{2p} \left(\frac{1}{2p-1} - \frac{1}{2p}\right)\right]$$

$$= 2\pi a \left[\log|a| + \sum_{p=1}^{\infty} \frac{(2p-2)!}{(p!)^2} \left(\frac{b}{2a}\right)^{2p}\right] = 2\pi a \left[\log|a| + \frac{1}{4}\sum_{p=1}^{\infty} \frac{\left(p - \frac{3}{2}\right)! \left(\frac{b}{a}\right)^{2p}}{p!\,(p)\left(-\frac{1}{2}\right)!}\right]$$

$$= 2\pi a \left[\log|a| - \sum_{p=1}^{\infty} \frac{\binom{p-\frac{3}{2}}{p}}{2p} \left(\frac{b}{a}\right)^{2p}\right] = 2\pi a \left[\log|a| - \sum_{p=1}^{\infty} \frac{(-1)^p}{2p} \binom{\frac{1}{2}}{p} \left(\frac{b}{a}\right)^{2p}\right]$$

$$= 2\pi a \left[\log|a| - \int_{o}^{b/a} dx \; \frac{[\sqrt{1 - x^2} - 1]}{x}\right]$$

$$= 2\pi a \left[1 - \sqrt{1 - b^2/a^2} + \log \left(\frac{|a| + \sqrt{a^2 - b^2}}{2}\right)\right] \qquad ,$$

$$|a| > b \qquad , \qquad \text{(A4)}$$

where we have used the integral

$$\int_0^{2\pi} d\phi \ \cos\phi^{2p} = \frac{1}{2^{2p}} \int_0^{2\pi} d\phi \sum_{q=o}^{2p} \binom{2p}{q} e^{i(2p-2q)\phi}$$

$$= \frac{2\pi}{2^{2p}} \binom{2p}{p} \equiv \frac{2\pi(2p)!}{2^{2p}(p!)^2} \qquad , \tag{A5}$$

and the factorial doubling formula

$$(2z)! = \frac{2^{2z} z!(z-\frac{1}{2})!}{(-\frac{1}{2})!} \tag{A6}$$

For b > a, it is convenient to write the integral (153) in the form

$$\int_0^{2\pi} d\phi \ (a + b \cos\phi) \log|a + b \cos\phi|$$

$$= b \int_0^{2\pi} d\phi \ (\cos\phi - \cos\phi_1)[\log b + \log|\cos\phi - \cos\phi_1|]$$

$$= b \left[- 2\pi \cos\phi_1 \ \log b + \int_0^{2\pi} d\phi \ (\cos\phi - \cos\phi_1) \ \log\left|2 \sin\left(\frac{\phi+\phi_1}{2}\right) \sin\left(\frac{\phi-\phi_1}{2}\right)\right|\right]$$

$$= 2\pi a \log b + \frac{b}{2}\left[\int_0^{2\pi} d\phi \ (\cos(\phi-\phi_1)\cos\phi_1\right.$$

$$- \sin(\phi-\phi_1) \sin\phi_1 - \cos\phi_1) \log (1 - \cos(\phi-\phi_1))$$

$$\left. + \left(\phi_1 \to - \phi_1\right)\right] \tag{A7}$$

The integral of the terms proportional to $\sin(\phi \pm \phi_1)$ clearly vanishes, and the remaining terms are of the same form calculated in (A4) (with $b/a = \pm 1$ and an irrelevant translation of the angular variable). It follows that

$$\int_0^{2\pi} d\phi \; (a + b\cos\phi) \log|a + b\cos\phi| = 2\pi a[\log b + 1 - \log 2],$$

$$b > |a| \qquad , \tag{A7}$$

which completes the proof of (153).

Substitution of (152), (153) into (149), the transformation $\mu = w/M$, and the use of the definitions (103), (108) leads to the form (154). We now turn to the calculation of the integrals C_1-C_6. For C_1 we write

$$C_1 = \frac{1}{\sqrt{2}} \int_{-M}^{M} dw \; Y^{1/2}(w) = \frac{1}{\sqrt{2}\,\pi} \int_{-\infty}^{\infty} du \int_{-M}^{M} dw \frac{Y(w)}{u^2 + Y(w)} \quad , \tag{A8}$$

where we have assumed that an interchange of integrals is permissible. It will be recalled that $Y(w)$ is analytic in the lower half-plane and vanishes as $(-1/2w^2)$ at ∞. It follows that

$$\int_{-\infty}^{\infty} dw \frac{Y(w)}{u^2 + Y(w)} = 0, \; u \neq 0, \tag{A9}$$

and

$$\int_{-M}^{M} dw \frac{Y(w)}{u^2 + Y(w)} = - \int_{|w|>M} \frac{dw \; Y(w)}{u^2 + Y(w)}$$

$$= 2P \int_{M}^{\infty} \frac{dw[1 + 0(\frac{1}{M^2})]}{2w^2(u^2 - \frac{1}{2w^2})} \cong \frac{1}{\sqrt{2}|u|} \log \left| \frac{|u| + \frac{1}{\sqrt{2}\,M}}{|u| - \frac{1}{\sqrt{2}\,M}} \right| \quad , \tag{A10}$$

According to (A9), the trick employed in (A10) is not valid as $u \to 0$; however, the result (A10) correctly approaches 2M as $u \to 0$, and is thus valid for all u. Substitution of (A10) into (A8) gives

$$C_1 \cong \frac{1}{\pi}\int_0^\infty \frac{du}{u} \log\left|\frac{u+\frac{1}{\sqrt{2}\,M}}{u-\frac{1}{\sqrt{2}\,M}}\right| = \frac{1}{\pi}\int_0^\infty \frac{du}{u}\log\left|\frac{u+1}{u-1}\right|$$

$$= \frac{1}{\pi}\left[2\int_0^1 du \sum_{p=o}^\infty \frac{u^{2p}}{2p+1} + 2\int_1^\infty du \sum_{p=o}^\infty \frac{\left(\frac{1}{u}\right)^{2p+2}}{2p+1}\right] = \frac{4}{\pi}\sum_{p=o}^\infty \frac{1}{(2p+1)^2}$$

$$= \frac{4}{\pi}\left[\sum_{p=1}^\infty \frac{1}{p^2} - \sum_{p=1}^\infty \frac{1}{(2p)^2}\right] = \frac{3}{\pi}\sum_{p=1}^\infty \frac{1}{p^2} = \frac{\pi}{2} \qquad \text{(A11)}$$

The error in (A11) is $O(1/M^2)$, as will be the case in the remainder of our calculations unless otherwise stated (of course, terms which behave as $M^n e^{-M^2}$ vanish much more strongly for large M, and will always be neglected).

The integral in (156) is easy to calculate since the real part of Y is even and the imaginary part odd. One finds

$$C_2 = \frac{2i}{\pi}\int_{-M}^{M} dw\ w\ Y(w) = -\frac{2}{\pi}\int_{-M}^{M} dw\ d\ Y_2(w) = \frac{2}{\sqrt{\pi}}\int_{-M}^{M} dw\ w^2\ {}^{-w^2}$$

$$= \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} dw\ w^2\ e^{-w^2} + O(M e^{-M^2}) \approx 1 \qquad \text{(A12)}$$

Similarly

$$C_3 = -\int_{-M}^{M} dw\,|w|\ Y(w) = -2\int_0^M dw\ w\ Y_1(w) = -\frac{2}{\sqrt{\pi}}\int_0^M dw\ w\ P\int_{-\infty}^{\infty}\frac{du\ u\,e^{-u^2}}{u-w}$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^M dw\ P\int_{-\infty}^{\infty}\frac{du\ u^2 e^{-u^2}}{u-w} = +\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\ u^2 e^{-u^2}\ \log\left|\frac{M-u}{u}\right|$$

$$= \log M - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\ u^2 \log|u| e^{-u^2} + \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\ u^2 \log\left|1-\frac{u}{M}\right| e^{-u^2} \qquad \text{(A13)}$$

where we have used the definition of Y. The last integral in (A13) may easily be shown to be $0(1/M^2)$. In fact, if

$$g(M) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\ u^2 \log\left|1-\frac{u}{M}\right| e^{-u^2} \quad , \qquad \text{(A14)}$$

one has

$$g(\infty) = 0 \quad ,$$

$$\frac{\partial g}{\partial M} = -\frac{2}{\sqrt{\pi}\ M}\ P\int_{-\infty}^{\infty}\frac{du\ u^3 e^{-u^2}}{u-M} = -\frac{1}{M} - \frac{2M}{\sqrt{\pi}}\ P\int_{-\infty}^{\infty}\frac{du\ u\ e^{-u^2}}{u-M}$$

$$= \frac{3}{2M^3}\left(1 + 0\left(\frac{1}{M^2}\right)\right) \quad , \qquad \text{(A15)}$$

where we have used the well known asymptotic form

$$\frac{1}{\sqrt{\pi}}\ P\int_{-\infty}^{\infty}\frac{du\ u\ e^{-u^2}}{u-M} = -\frac{1}{2M^2}\left(1 + \frac{3}{2M^2} + 0\left(\frac{1}{M^4}\right)\right) \qquad \text{(A16)}$$

Thus

$$g(M) = -\frac{3}{4M^2}\left(1 + 0\left(\frac{1}{M^2}\right)\right) \quad , \qquad \text{(A17)}$$

and

$$C_3 \stackrel{\sim}{=} \log M - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\ u^2 \log|u| e^{-u^2} \qquad \text{(A18)}$$

Inasmuch as the second term of (A18) will cancel with similar terms from other C_i's, we will not attempt to do the integral.

Turning to (158), we write

$$C_4 = -\frac{i}{\pi}\int_{-M}^{M} dw\ w\ Y(w)\ [\log|Y(w)| + i \arg Y(w)]$$

$$= -\frac{i}{\pi}\int_{-M}^{M} dw\ w\ Y(w)\ \log Y(w) = -\frac{i}{\pi}\int_{-M}^{M} dw\ w\ Y(w)\int_{o}^{\infty} dx \left[\frac{1}{x+1} - \frac{1}{x+Y}\right] . \tag{A19}$$

But (cf. A12)

$$\int_{-M}^{M} dw\ w\ Y(w) \approx - i\pi/2 \tag{A20}$$

Because

$$Y(-w) = Y^*(w) \qquad , \tag{A21}$$

one has

$$\int_{-M}^{M} dw\ \frac{wY(w)}{x+Y(w)} = -\frac{x}{2}\int_{-M}^{M} dw\ w \left[\frac{1}{x+Y} - \frac{1}{x+Y^*}\right]$$

$$= -\frac{x}{2}\lim_{\epsilon\to o}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} dw\ w \left[\left(\frac{1}{x+Y(w)} - \frac{1}{x - \frac{1}{2w^2}}\right) - \left(\frac{1}{x+Y^*(w)} - \frac{1}{x - \frac{1}{2w^2}}\right)\right]$$

$$- 2ix\int_{M}^{\infty} \frac{dw\ w\ Y_2(w)}{[x+Y_1(w)]^2 + Y_2(w)^2} \tag{A22}$$

Inasmuch as $Y(w)$ is analytic in the lower half-plane and Y^* in the upper, the first term can be evaluated by contour integration. For the second term we note that, for large w, Y_2 is very small, and negative, and Y_1 is approximately $(-\frac{1}{2w^2})$, so that

$$\frac{Y_2(w)}{(x+Y_1)^2 + Y_2^{\,2}} \overset{\sim}{=} - \lim_{\epsilon\to o} \frac{\epsilon}{(x - \frac{1}{2w^2})^2 + \epsilon^2} = - \pi\ \delta(x - \frac{1}{2w^2}) \quad . \tag{A23}$$

It follows that

$$\int_{-M}^{M} \frac{dw\ w\ Y(w)}{x+Y(w)} \approx - \frac{\pi i}{2x}\ S\left(x - \frac{1}{2M^2}\right) \tag{A24}$$

While the contour integration technique as applied here is not strictly valid for $x = 0$, it will be observed that (A24) vanishes as $x \to 0$ as it must.

Substituting (A24) into (A19), one finds

$$C_4 \overset{\sim}{=} \log\ (\sqrt{2}\ M) \tag{A25}$$

We now turn to the more difficult angle dependent integrals C_5, C_6, which can be only partially evaluated for general θ. From (160),

$$C_6(\theta) = - \frac{2}{\pi} \int_{-M}^{M} dw\ Y(w)\ \Big[\sqrt{M^2\sin^2\theta - w^2}$$

$$S(M\sin\theta - |w|) - i(\text{sgn } w)\sqrt{w^2 - M^2\sin^2\theta}\ S(|w| - M\sin\theta)\Big]$$

$$= - \frac{2}{\pi} \lim_{\epsilon\to o} \int_{-M+i\epsilon}^{M+i\epsilon} dw\ Y(w)(M^2\sin^2\theta - w^2)^{1/2} \tag{A26}$$

where $(M^2\sin^2\theta - w^2)^{1/2}$ is defined as follows

$$M\sin\theta - w = R_1 e^{i\phi_1},\ - \pi < \phi_1 < \pi,\quad M\sin\theta + w = R_2 e^{i\phi_2},\ - \pi < \phi_2 < \pi,$$

$$(M^2\sin^2\theta - w^2)^{1/2} = \sqrt{R_1 R_2}\ e^{i(\phi_1+\phi_2)/2} \tag{A27}$$

The integral in (A26) may be broken up as follows

$$C_6(\theta) = -\frac{2}{\pi}\left\{\int_{-M}^{M} dw\ Y(w)\ M\sin\theta + \int_{-M+i\epsilon}^{M+i\epsilon} dw[(M^2\sin^2\theta-w^2)^{1/2} - M\sin\theta]\ \left(-\frac{1}{2w^2}\right)\right.$$

$$\left. + \int_{-M+i\epsilon}^{M+i\epsilon} dw[(M^2\sin^2\theta-w^2)^{1/2} - M\sin\theta]\ [Y(w) + \frac{1}{2w^2}]\right\} \quad . \qquad \text{(A28)}$$

The first two integrals are straightforward, taking into account the symmetry of $Y(w)$, $(M^2\sin^2\theta - w^2)^{1/2}$. In fact

$$\int_{-M}^{M} dw\ Y(w) = \int_{-M}^{M} dw\ Y_1(w) = 2\int_{o}^{M} dw\left[1 - 2we^{-w^2}\int_{o}^{w} du\, e^{u^2}\right]$$

$$= 2\, e^{-w^2}\int_{o}^{w} du\, e^{u^2}\Bigg|_{w=0}^{w=M} = \frac{1}{M}\left(1 + 0(\frac{1}{M^2})\right) \quad , \qquad \text{(A29)}$$

where we have employed an obvious integration by parts. In the second term we have the integral

$$\lim_{\epsilon\to o}\int_{-M+i\epsilon}^{M+i\epsilon}\frac{dw}{w^2}\ [(M^2\sin^2\theta - w^2)^{1/2} - M\sin\theta] = 2\int_{o}^{M\sin\theta}\frac{dw}{w^2}\ [\sqrt{M^2\sin^2\theta-w^2} - M\sin\theta]$$

$$- 2M\sin\theta\int_{M\sin\theta}^{M}\frac{dw}{w^2} = 2\sin\theta - \pi \qquad \text{(A30)}$$

The third term of (A28) may be written as

$$\lim_{\epsilon\to o}\left\{\int_{-\infty+i\epsilon}^{\infty+i\epsilon} dw\ [(M^2\sin^2\theta - w^2)^{1/2} - M\sin\theta]\ \left[Y(w) + \frac{1}{2w^2}\right]\right.$$

$$-\left(\int_{-\infty+i\epsilon}^{-M+i\epsilon} + \int_{M+i\epsilon}^{\infty+i\epsilon}\right) dw[(M^2\sin^2\theta - w^2)^{1/2} - M\sin\theta]\left[Y(w) + \frac{1}{2w^2}\right]\Bigg\}.$$

The second term in the curly bracket is readily shown to be $0(M^{-2})$. The first term may be expressed as

$$\int_c dw\ [(M^2\sin^2\theta - w^2)^{1/2} - M\sin\theta]\left[Y(w) + \frac{1}{2w^2}\right]$$

$$+ \int_{-\infty-i\epsilon}^{\infty-i\epsilon} dw\ [(M^2\sin^2\theta - w^2)^{1/2} - M\sin\theta]\left[Y(w) + \frac{1}{2w^2}\right] \quad ,$$

where c is a path enclosing the branch cuts (extending from $\pm$ $M\sin\theta$ to $\pm\infty$) as depicted in Fig. 5. But the integrand in the second term is analytic in the lower half-plane and vanishes as w^{-3} there; then the path may be closed in the lower half-plane and a null contribution is obtained. The branch cut contribution may be simplified by symmetry considerations, and one finds

$$\lim_{\epsilon\to o} \int_{-M+i\epsilon}^{M+i\epsilon} dw\ [(M^2\sin^2\theta - w^2)^{1/2} - M\sin\theta]\left[Y(w) + \frac{1}{2w^2}\right]$$

$$\cong -4\sqrt{\pi} \int_{M\sin\theta}^{\infty} dw\ w\sqrt{w^2-M^2\sin^2\theta}\, e^{-w^2} \tag{A31}$$

Employing (A29), (A30), (A31) in (A28), one finds

$$C_6(\theta) \cong -1 + \frac{8}{\sqrt{\pi}} \int_{M\sin\theta}^{\infty} dw\ w\sqrt{w^2-M\sin^2\theta}\, e^{-w^2} \tag{A32}$$

In a quite similar fashion, one may write (159) as

$$C_5(\theta) = -\frac{2i}{\pi}\left\{\int_{-M}^{M} dw\ w\ Y(w)\ \log|w| + \lim_{\epsilon\to o}\int_{-M+i\epsilon}^{M+i\epsilon} dw\ w\ Y(w)\log\left[1+\frac{i(M^2\sin^2\theta-w^2)^{\frac{1}{2}}}{w\cos\theta}\right]\right\}$$

$$= -\frac{2}{\sqrt{\pi}}\int_{-M}^{M} dw\ w^2\log|w|e^{-w^2} - \frac{2i}{\pi}\int_{-M}^{M} dw\ w\ Y(w)\left[\log\frac{M\tan\theta}{|w|}+\frac{i\pi}{2}\ \text{sgn}\ w\right]$$

$$+\frac{i}{\pi}\lim_{\epsilon\to o}\int_{-M+i\epsilon}^{M+i\epsilon}\frac{dw}{w}\log\left(\frac{(M^2\sin^2\theta - w^2)^{1/2} - iw\cos\theta}{M\sin\theta}\right)$$

$$-\frac{2i}{\pi}\lim_{\epsilon\to o}\left[\int_{-M+i\epsilon}^{M+i\epsilon} - \int_{-M-i\epsilon}^{M-i\epsilon}\right] dw\ w\left[Y(w)+\frac{1}{2w^2}\right]$$

$$\log\left(\frac{(M^2\sin^2\theta - w^2)^{1/2} - iw\cos\theta}{M\sin\theta}\right) + \frac{2i}{\pi}\lim_{\epsilon\to o}\int_{-M-i\epsilon}^{M-i\epsilon} dw\ w\left[Y(w)+\frac{1}{2w^2}\right]$$

$$\log\left(\frac{(M^2\sin^2\theta - w^2)^{1/2} - iw\cos\theta}{M\sin\theta}\right) \tag{A33}$$

The last term in (A33) is readily shown to be $0(M^{-2})$, and some of the remaining integrals may be evaluated with the aid of (A12), (A13), (A18). One finds

$$C_5(\theta) \stackrel{\sim}{=} -\log\left(\frac{M^2\sin^2\theta}{\cos\theta}\right) + \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} dw\ w^2\log|w|e^{-w^2}$$

$$-\frac{4}{\sqrt{\pi}}\int_{M\sin\theta}^{M} dw\ w^2 e^{-w^2}\ \log\left(\frac{w+\sqrt{w^2-M^2\sin^2\theta}}{w-\sqrt{w^2-M^2\sin^2\theta}}\right)$$

$$+\frac{2}{\pi}\int_{0}^{M\sin\theta} \frac{dw}{w}\ \arctan\left(\frac{w\cos\theta}{\sqrt{M^2\sin^2\theta-w^2}}\right) \qquad \text{(A34)}$$

The last integral in (A34) may be evaluated exactly. Introducing a new variable

$$v = \arctan\left(\frac{w\cos\theta}{\sqrt{M^2\sin^2\theta-w^2}}\right) \qquad ,$$

one has

$$\frac{2}{\pi}\int_{0}^{M\sin\theta} \frac{dw}{w}\ \arctan\left(\frac{w\cos\theta}{\sqrt{M^2\sin^2\theta-w^2}}\right) = \frac{2\cos^2\theta}{\pi}\int_{0}^{\pi/2} dv\ v\ \frac{\cot\ v}{\cos^2\theta+\sin^2\theta\sin^2 v}$$

$$= \frac{1}{\pi}\left[v\ \log\left(\frac{\sin^2 v}{\cos^2\theta+\sin^2\theta\sin^2 v}\right)\Bigg|_{0}^{\frac{\pi}{2}} - \int_{0}^{\pi/2} dv\ \log\left(\frac{\sin^2 v}{\cos^2\theta+\sin^2\theta\sin^2 v}\right)\right]$$

$$= \log\ (1+\cos\theta) \qquad \text{(A35)}$$

Thus

$$C_5(\theta) = -\log\left[\frac{M^2\sin^2\theta}{\cos\theta(1+\cos\theta)}\right] + \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} dw\ w^2\log|w|e^{-w^2} + A(\theta), \qquad \text{(A36)}$$

where $A(\theta)$ is given by (162). Substituting (A11), (A12), (A18), (A25), (A32), and (A36) in (154), one obtains (161).

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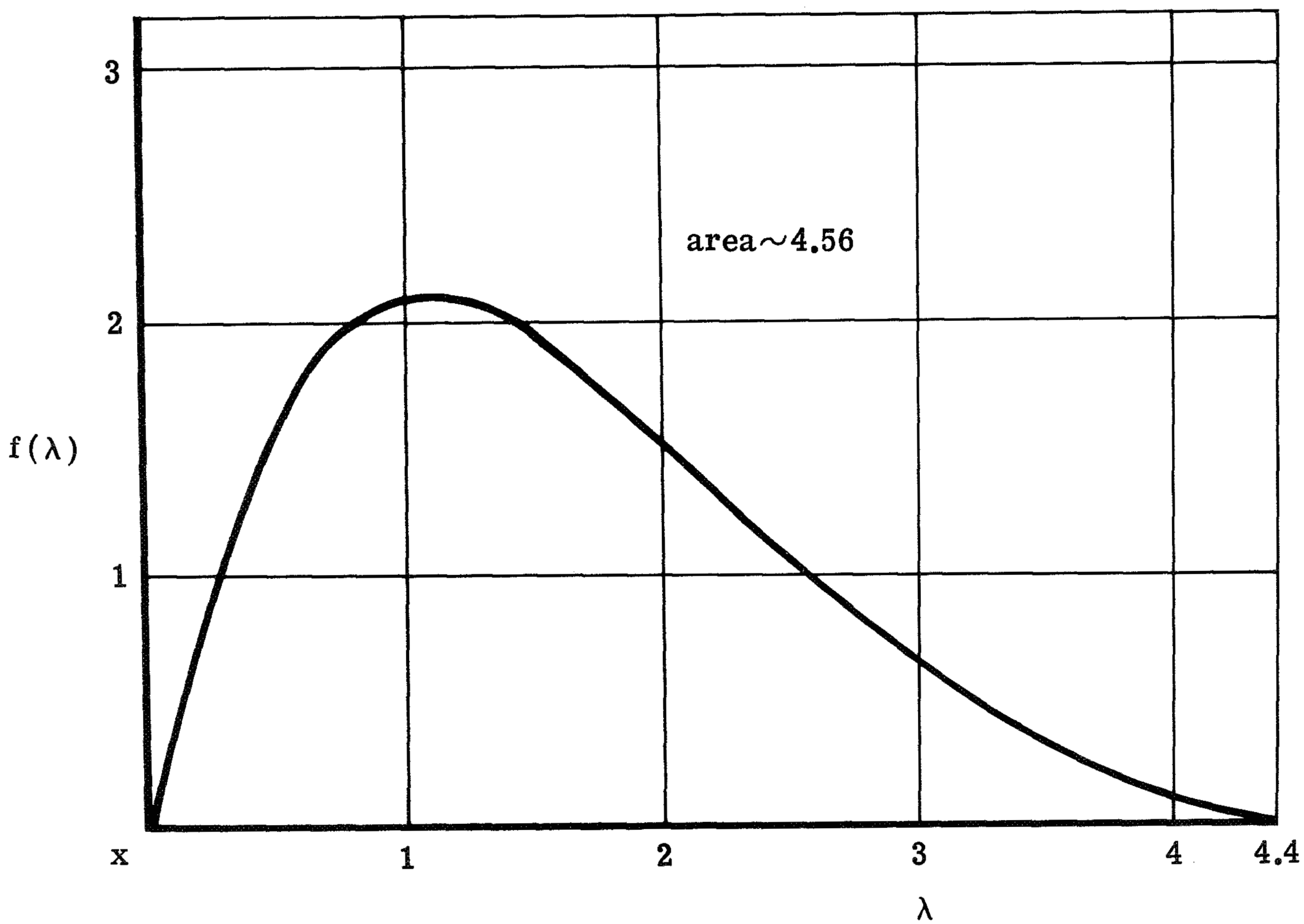


Fig. 1 - Profile of O(u/V) charge density correction.

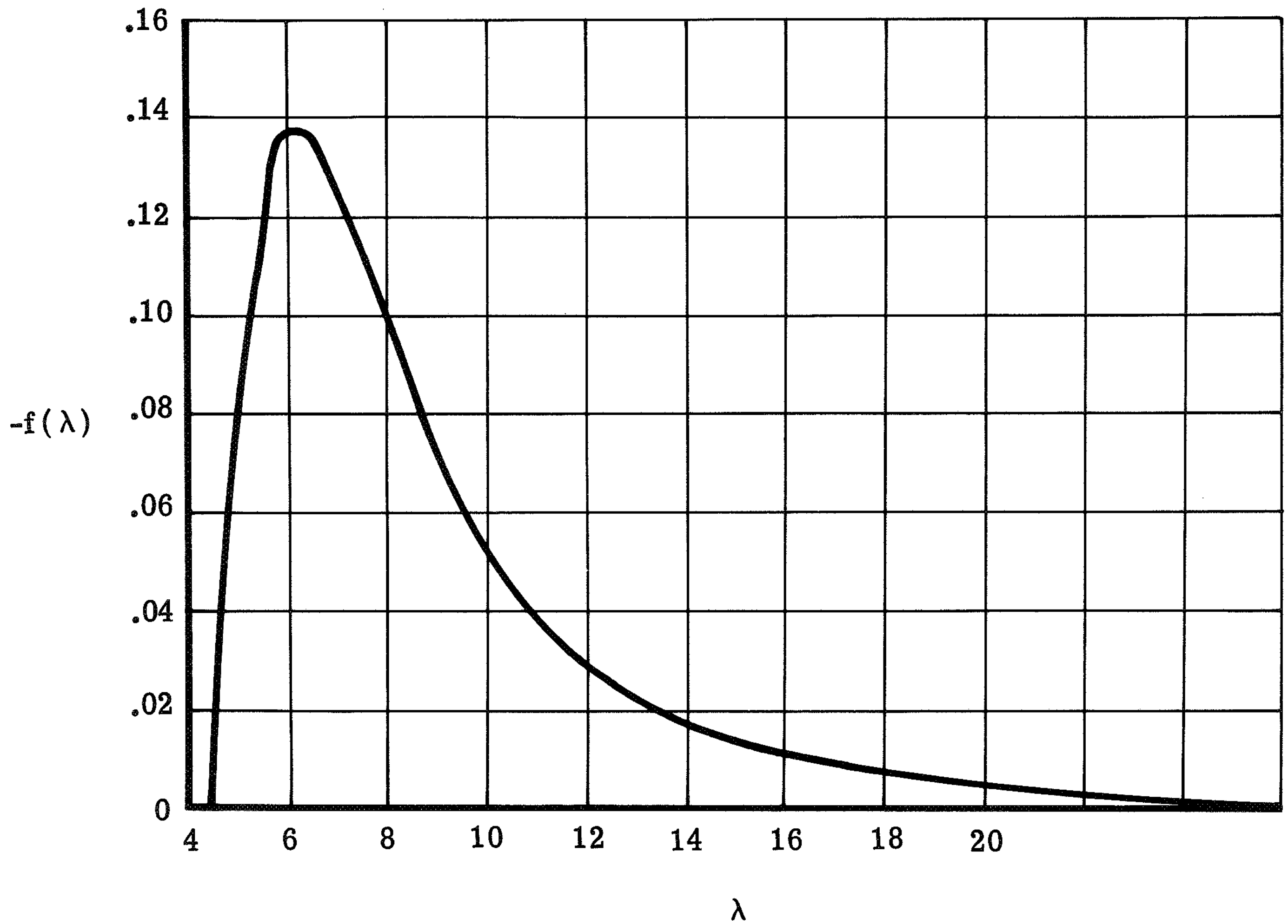


Figure 1 Continued

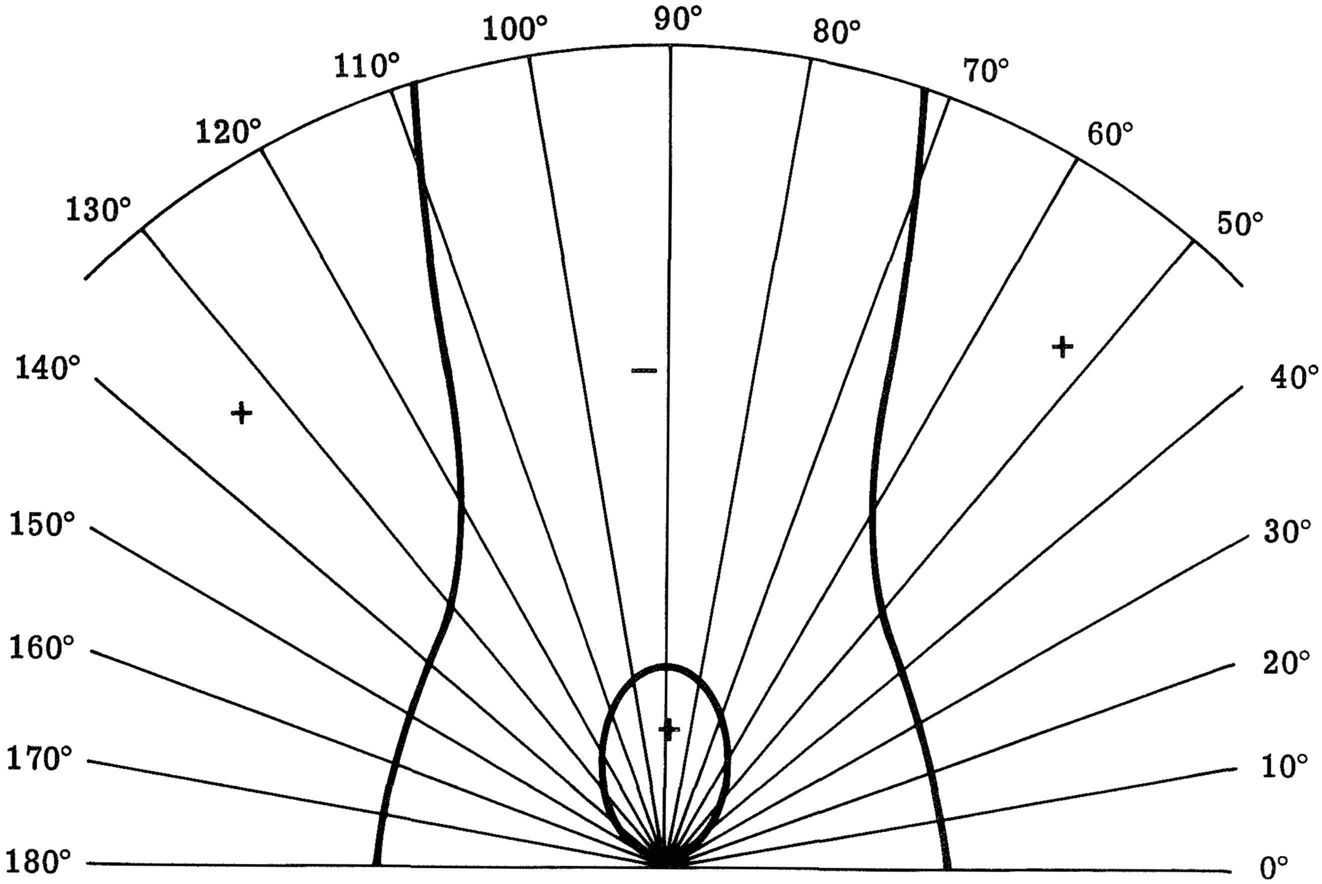


Fig. 2(a) - Distribution of $O(M^2)$ correction to charge density.

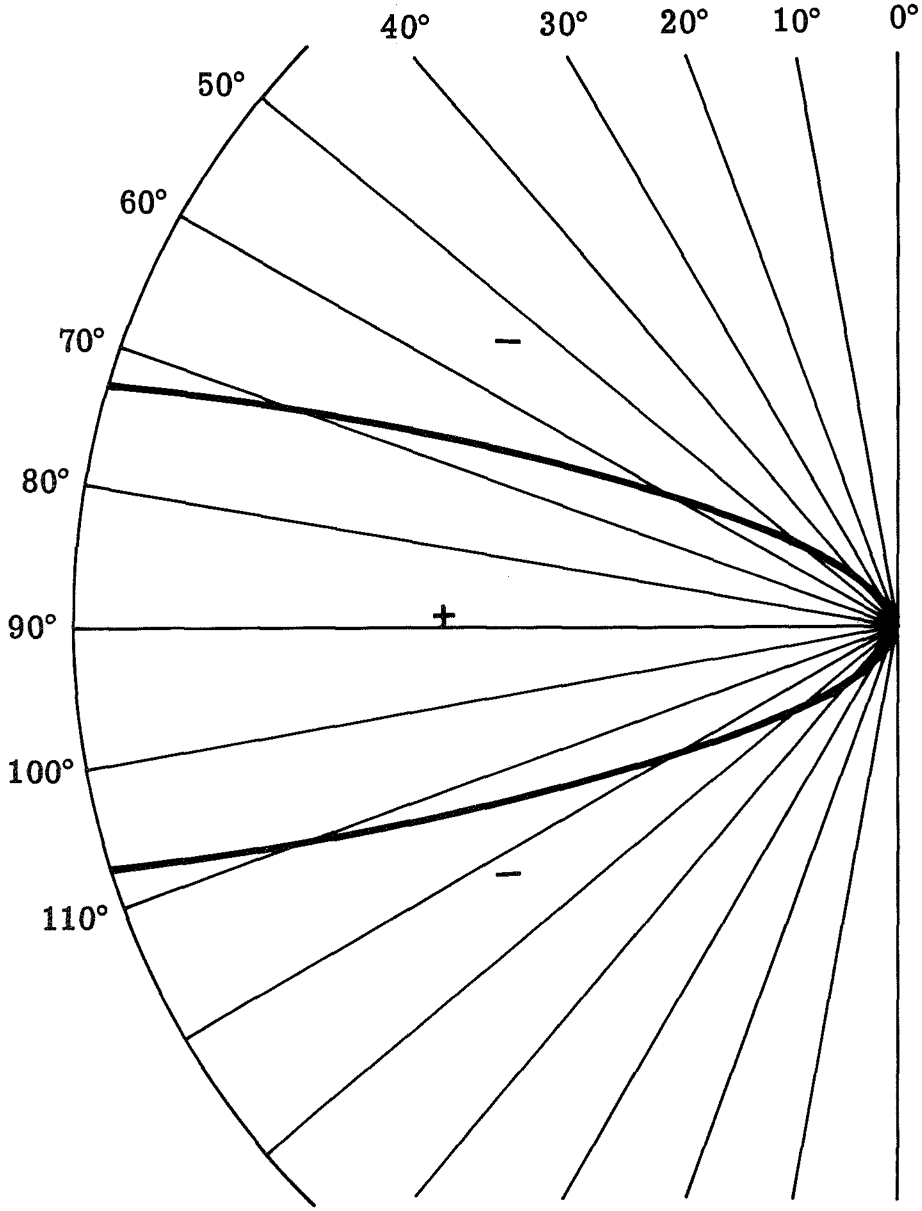
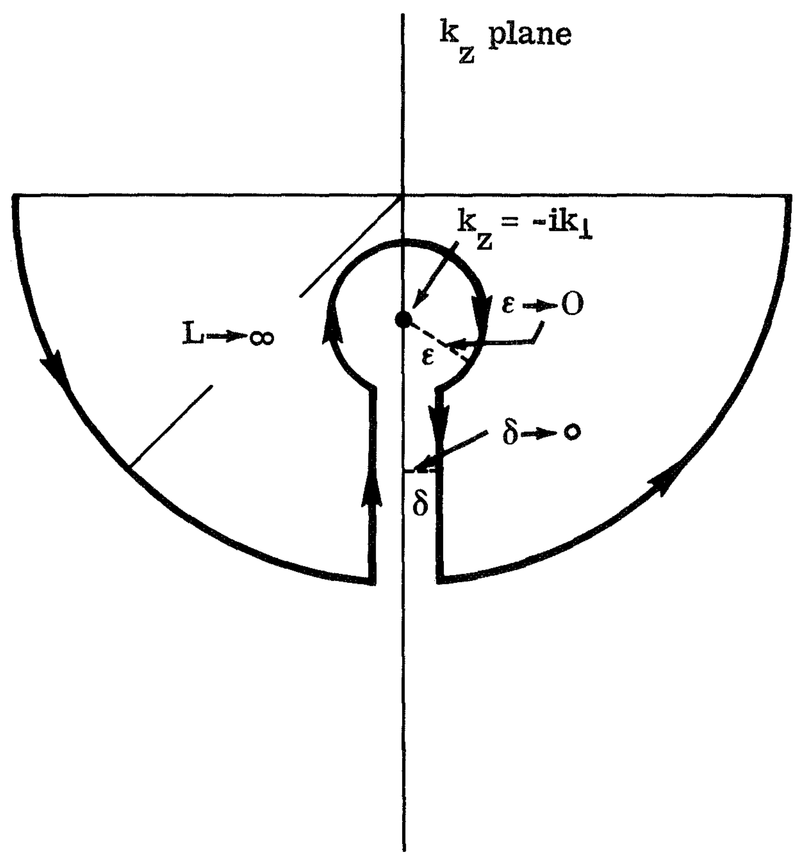


Fig. 2(b) - Distribution of $0(M^2)$ correction to charge density when $IM\Delta^-$ is neglected.



k_z plane
$k_z = -ik_\perp$
$\varepsilon \to 0$
ε
$L \to \infty$
$\delta \to 0$
δ

Figure 3

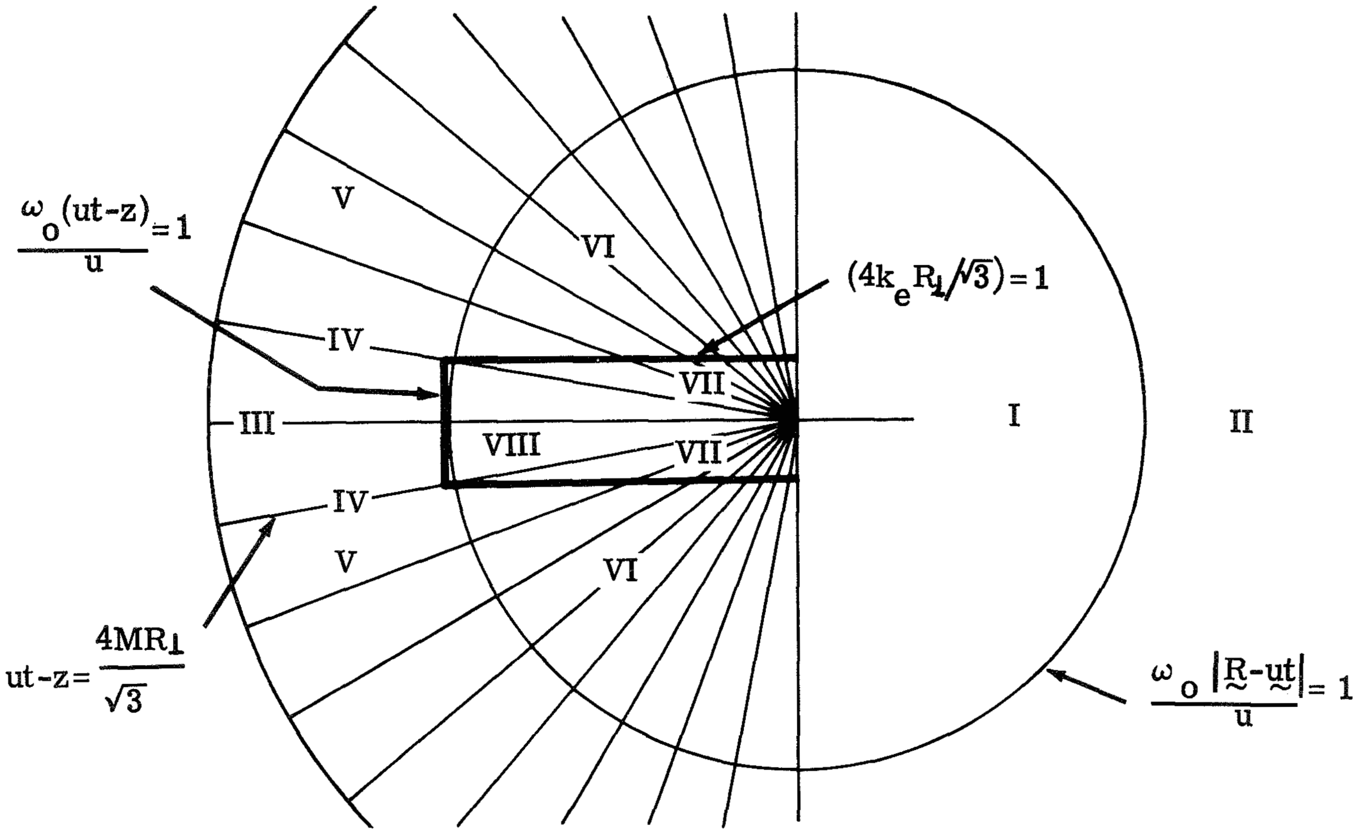


Fig. 4 - Regions for which the fast particle results are summarized in Table I.

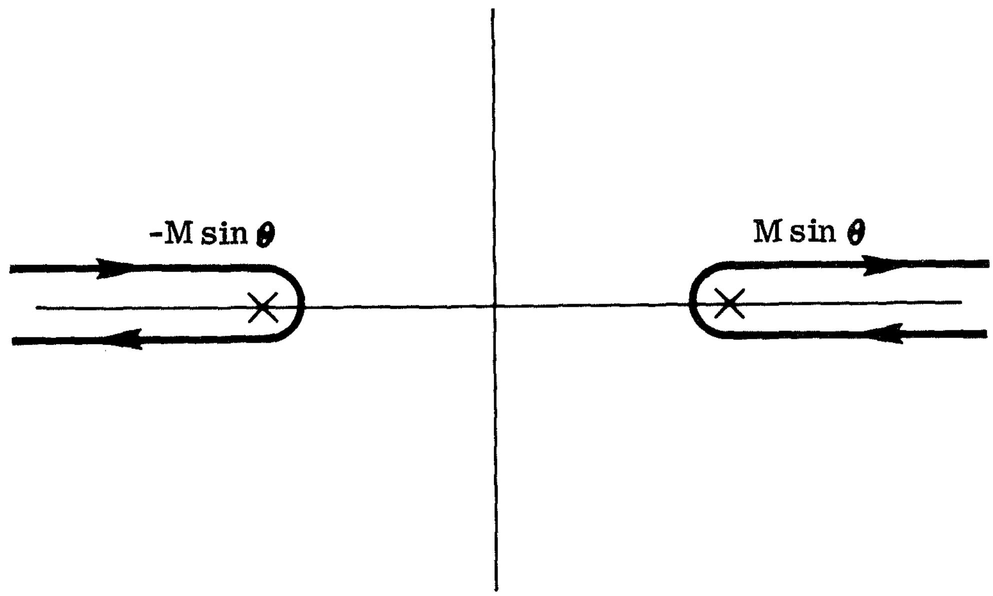


Figure 5